NO CALCULATOR ALLOWED

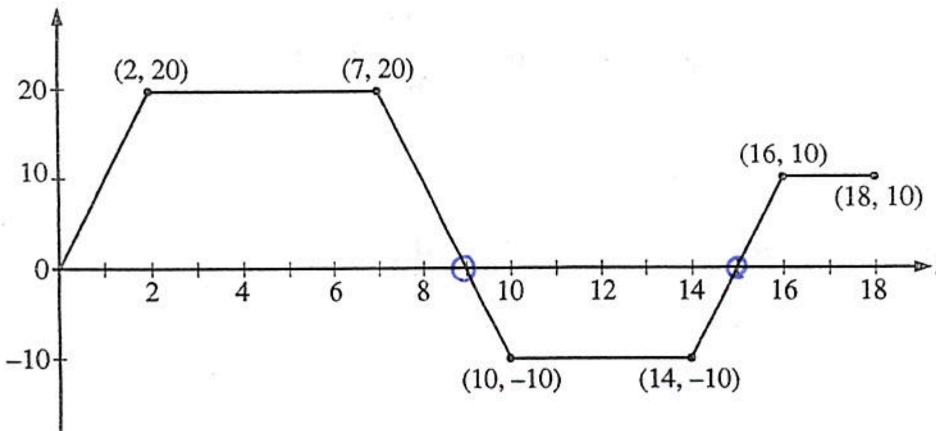
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

4. A squirrel starts at building A at time $t = 0$ and travels along a straight, horizontal wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

↪ velocity change signs

@ $t=9$ and $t=15$, squirrel changes direction

b/c $v(t)$ changes signs @ $t=9$ + $t=15$

1pt – t-value

1pt – reason

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at that time?

↪ max distance → $\int_0^t v(t)\,dt$

$x\nearrow\ x\searrow\ x\nearrow$

v: $+$ $-$ $+$ (0, 9, 15, 18)

↓ rel. max

$x(0) = 0$

$x(9) = \int_0^9 v(t)\,dt = \frac{1}{2}(9+5)(20)$

$= 140$

1pt – check crit pt + end pts

$x(18) = \int_0^{18} v(t)\,dt = 140 + \frac{1}{2}(6+4)(-10) + \frac{1}{2}(3+2)(10)$

$= 115$

1pt – answers

Squirrel farthest @ $t=9$. Squirrel is 140 units from building A.

Continue problem 4 on page 11.

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

$\hookrightarrow \int |v(t)|\,dt$

total distance $= \int_0^{18} |v(t)|\,dt$

$= 140 + 50 + 25$ ← 1pt- answer

$= 215$

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

slope of line (7,10)

line from (7,10)

$\hookrightarrow \int v(t)$

$a(t) = v'(t)$

$= \dfrac{v(7) - v(10)}{7 - 10}$

$= \dfrac{20 - (-10)}{-3}$

$a(t) = -10$ 1pt - $a(t)$

$y - y_1 = m(x - x_1)$

$y - 20 = -10(x - 7)$

$y = -10(x-7) + 20$

$v(t) = -10(t-7) + 20$ $-10t + 70 + 20$ $-10t + 90$ 1pt - $v(t)$

$x(t) = \int (-10t + 90)\,dt$

$x(t) = -5t^2 + 90t + C$

$x(9) = -5(9)^2 + 90(9) + C$ use $x(9)$ b/c already know $x(9) = 140$ from part (b)

$140 = -405 + C$

$-265 = C$

$x(t) = -5t^2 + 90t - 265$ 1pt - $x(t)$

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5. Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1+4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

$g'(x) = 0$ or DNE and check endpts

$$g'(x) = \frac{(1+4x^2)(4) - 4x(8x)}{(1+4x^2)^2}$$

2pts - $g'(x)$

$$0 = \frac{4-16x^2}{(1+4x^2)^2}$$

$$4 - 16x^2 = 0$$
$$4 = 16x^2$$
$$\frac{1}{4} = x^2$$
$$\pm\frac{1}{2} = x$$
$$x = \frac{1}{2}$$

1pt - crit #

g ↗ ↘

g': $+$ on $(0, \frac{1}{2})$, $-$ on $(\frac{1}{2}, \infty)$; test points $\frac{1}{4}$, 1

abs max @ $x = \frac{1}{2}$ b/c $g' > 0$ on $(0, \frac{1}{2})$ and $g' < 0$ on $(\frac{1}{2}, \infty)$

abs max is $g(\frac{1}{2}) = \frac{4(\frac{1}{2})}{1+4(\frac{1}{2})^2} = \frac{2}{2} = 1$

1pt - answers

1pt - reason

no abs min for g on $(0, \infty)$

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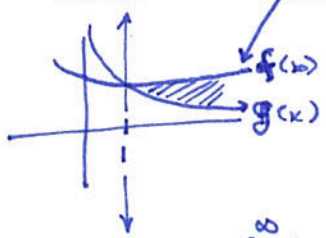
Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

below → under $f(x)$ and on top of $g(x)$

$f(x)$ on top, $g(x)$ on bottom



$$\text{Area} = \int_1^{\infty} (f(x) - g(x))\,dx$$

$$= \int_1^{\infty} \left(\frac{1}{x} - \frac{4x}{1+4x^2}\right) dx$$ 1pt-integral

$$= \lim_{a\to\infty}\left(\int_1^a \frac{1}{x}\,dx - \int_1^a \frac{4x}{1+4x^2}\,dx\right)$$

$u = 1+4x^2$

$\frac{du}{dx} = 8x$

$\frac{du}{8x} = dx$

$u(a) = 1+4a^2$

$u(1) = 5$

$$= \lim_{a\to\infty}\left(\ln|x|\Big|_1^a - \int_5^{1+4a^2} \frac{4x}{u}\cdot\frac{du}{8x}\right)$$

$$= \lim_{a\to\infty}\left(\ln a - \ln 1 - \frac{1}{2}\int_5^{1+4a^2} \frac{1}{u}\,du\right)$$

$$= \lim_{a\to\infty}\left(\ln a - \left(\frac{1}{2}\ln|u|\right)\Big|_5^{1+4a^2}\right)$$ ← 2pt - antiderivative

$$= \lim_{a\to\infty}\left(\ln a - \left(\frac{1}{2}\ln(1+4a^2) - \frac{1}{2}\ln 5\right)\right)$$

$$= \lim_{a\to\infty}\left(\ln a - \frac{1}{2}\ln(1+4a^2) + \frac{1}{2}\ln 5\right)$$

$$= \lim_{a\to\infty}\left(\ln\left(\frac{\sqrt{5}\,a}{\sqrt{1+4a^2}}\right)\right)$$

$$= \ln\left(\frac{\sqrt{5}}{\sqrt{4}}\right)$$

$$= \ln\left(\frac{\sqrt{5}}{2}\right) \quad \text{or} \quad \frac{1}{2}\ln\left(\frac{5}{4}\right)$$ 1pt-answer

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6. The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

$|x-a| < 1$
$-\# < x < \#$

* Ratio Test *

$\lim_{n\to\infty} \frac{(-1)^{n+1}(2x)^{n+1}}{n+1-1} \cdot \frac{n-1}{(-1)^n(2x)^n}$

1pt - set up Ratio Test

1pt - evaluates limit

$= \lim_{n\to\infty} -1(2x)\left(\frac{n-1}{n}\right)$

$= -2x$

$|-2x| < 1$

$|2x| < 1$

$|x| < \frac{1}{2}$

← 1pt - radius of convergence

interval of convergence: $-\frac{1}{2} < x \leq \frac{1}{2}$

← 1pt - analysis of endpts + answer correct

check endpts:

$x = -\frac{1}{2}$

$\sum_{n=2}^{\infty} \frac{(-1)^n(2\cdot -\frac{1}{2})^n}{n-1}$

$\sum_{n=2}^{\infty} \frac{1}{n-1}$

diverges p-test where $p = 1$

so, $x = -\frac{1}{2}$ not included

$x = \frac{1}{2}$

$\sum_{n=2}^{\infty} \frac{(-1)^n(2\cdot\frac{1}{2})^n}{n-1}$

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$

1pt - considers endpts

* AST *

(1) $\frac{1}{n-1} > 0$ for $n \geq 2$ ✓

(2) $\frac{1}{n-1} > \frac{1}{n}$ ✓ b/c $n > n-1$

(3) $\lim_{n\to\infty} \frac{1}{n-1} = 0$ ✓

converges by AST so $x = \frac{1}{2}$ included

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Continue problem 6 on page 15.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

$$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} + \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 + \cdots + \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$y' = 8x - 12x^2 + \frac{16}{3}(4x^3) + \cdots + \frac{(-1)^n \cdot n(2x)^{n-1} \cdot 2}{n-1} + \cdots$$

1pt- y' series

$$xy' = 8x^2 - 12x^3 + \frac{16}{3}(4x^4) + \cdots + \frac{(-1)^n \cdot 2n(2x)^{n-1} \cdot x}{n-1}$$

1pt - xy' series

$$xy' - y = 4x^2 - 8x^3 + \cdots + \frac{(-1)^n \cdot 2n(2x)^{n-1} \cdot x}{n-1} - \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 8x^3 + \cdots + \frac{(-1)^n \cdot n \cdot (2x) \cdot (2x)^{n-1}}{n-1} - \frac{(-1)^n(2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 8x^3 + \cdots + \frac{(-1)^n \cdot n \cdot (2x)^n}{n-1} - \frac{(-1)^n(2x)^n}{n-1}$$

$$= 4x^2 - 8x^3 + \cdots + \frac{(-1)^n(2x)^n}{n-1}(n-1)$$

$$= 4x^2 - 8x^3 + \cdots + (-1)^n(2x)^n$$

1pt - xy' - y series

$$= 4x^2\underbrace{\left(1 - 2x + \cdots + (-1)^n(2x)^{n-2} + \cdots\right)}_{\sum_{n=2}^{\infty}(-1)^n(2x)^{n-2} = \sum_{n=0}^{\infty}(-1)^n(2x)^n}$$

geo series converges on $|r|<1$

~~abs~~ to $\frac{a_1}{1-r} = \frac{1}{1-(-2x)}$

$= \frac{1}{1+2x}$ when $|-2x|<1$

$|x| < \frac{1}{2}$

1pt - analysis of answer

$$\therefore\ xy' - y = 4x^2\left(\frac{1}{1+2x}\right) \text{ for } |x| < \tfrac{1}{2}$$