AB
(b) & (c)

3 3 3 3 3 3 3 3 3 3

NO CALCULATOR ALLOWED

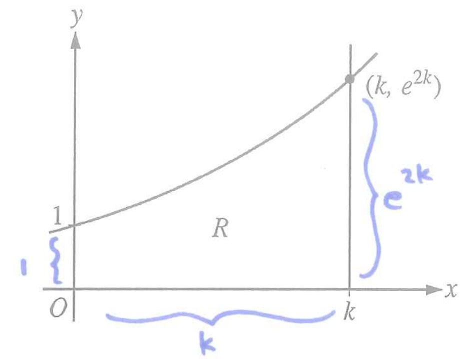
CALCULUS ~~AB~~ BC

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



3. Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.

(a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

line + line + line + arc length

Perimeter of R $= 1 + k + e^{2k} + \int_0^k \sqrt{1+(2e^{2x})^2}\,dx$

$f'(x) = e^{2x} \cdot 2$

1pt - $f'(x)$
1pt - integral
1pt - answer

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Continue problem 3 on page 9.

(b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

work in x's

VOLUME
in DISK

$$V = \pi \int_0^k (e^{2x})^2 \, dx$$

$$= \pi \int_0^k e^{4x} \, dx \qquad u = 4x \qquad \frac{du}{dx} = 4$$

$$= \pi \int_0^{4k} e^u \cdot \frac{du}{4} \qquad \frac{du}{4} = dx \qquad u(0) = 0 \qquad u(k) = 4k$$

$$= \frac{\pi}{4} (e^u) \Big|_0^{4k}$$

$$= \frac{\pi}{4}(e^{4k} - e^0) = \frac{\pi}{4}(e^{4k} - 1)$$

1pt - integrand

1pt - limits

1pt - antiderivative

1pt - answer

(c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

$$V = \frac{\pi}{4}(e^{4k} - 1)$$

$$\frac{dV}{dt} = \frac{\pi}{4}\left(e^{4k} \cdot 4 \frac{dk}{dt} - 0\right)$$

$$\frac{dV}{dt}\Big|_{k=1/2} = \frac{\pi}{4}\left(e^{4(1/2)} \cdot 4\left(\tfrac{1}{3}\right)\right)$$

$$\frac{dV}{dt}\Big|_{k=1/2} = \frac{\pi}{3} e^2$$

1pt - chain rule one ~~(dk w/ respect to time)~~

1pt - answer

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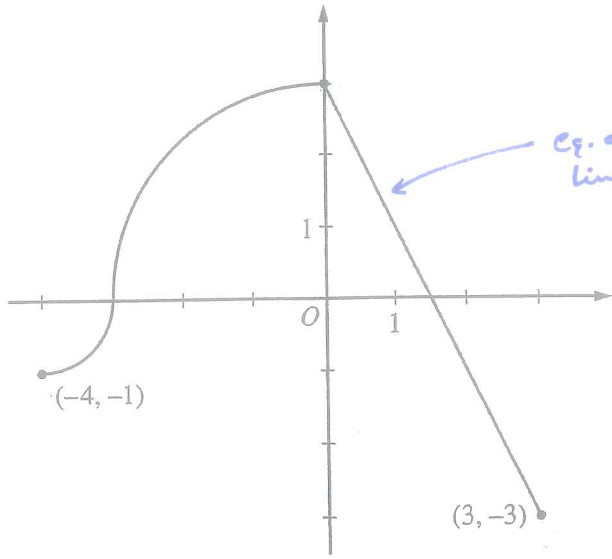
AB

NO CALCULATOR ALLOWED



eq. of line = $y - y_1 = m(x - x_1)$

$y - 3 = -\frac{6}{3}(x - 0)$

$y = -2x + 3$

$-2 = -2x + 3$

Graph of f

4. The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

$g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt$

$= -6 + -\frac{1}{4}(\pi)(3)^2$

$= -6 - \frac{9}{4}\pi$

1pt $g(-3)$

1pt - $g'(x)$

1pt - $g'(-3)$

$g'(x) = \frac{d}{dx}(2x + \int_0^x f(t)\,dt)$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3)$

$= 2 + 0$

$g'(-3) = 2$

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

rel. max $g' = 0$

$g'(x) = 2 + f(x) = 0$

$f(x) = -2$

$-2x + 3 = -2$

$-2x = -5$

$x = 5/2$

g ↗ ↘

g' + −

④ 5/2 ③

1pt - $g'(x) = 0$

1pt - crit #

1pt - answer

abs max @ $x = 5/2$ b/c ~~g' changes~~ $g' > 0$ on $(-4, 5/2)$ and $g' < 0$ on $(5/2, 3)$

no need to check endpts b/c g' doesn't change signs again

:)

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Continue problem 4 on page

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

g'' changes signs

$f' = g''$ $+ \quad + \quad -$ at -3, 0

$g' = 2 + f(x)$

$g''(x) = f'(x)$

g has pt of inf @ $x = 0$

b/c g'' changes signs @ $x = 0$

1pt - answer w/reason

$\frac{f(b)-f(a)}{b-a}$

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

avg rate change $= \frac{f(3) - f(-4)}{3 - -4}$

$= \frac{-3 - -1}{7} = \frac{-2}{7}$

1pt - avg rate change

MVT → f cont on $(-4, 3)$? yes

f diff'able on $(-4, 3)$? no

f is not diff'able @ $x = 0$ and $x = -3$,

∴ MVT does not apply.

so the statement cannot contradict MVT.

1pt - explain

NO CALCULATOR ALLOWED

5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $\boxed{t = 0}$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$y - y_1 = m(x - x_1)$

(t, W)
$(0, 1400)$

$y - 1400 = 44(t - 0)$

$\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(1400 - 300)$
$= \frac{1}{25}(1100)$
$= \frac{1100}{25} = 44$

$y - 1400 = 44\left(\frac{1}{4} - 0\right)$

$y - 1400 = 11$

$y = 1411$

$W\left(\frac{1}{4}\right) = 1411$ tons

1pt - $\frac{dW}{dt}$ @ $t = 0$

1pt - answer

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

$\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\frac{d^2W}{dt^2} = \frac{1}{25}\left(1\frac{dW}{dt}\right)$

$= \frac{1}{25}\frac{dW}{dt}$

$= \frac{1}{25}\left(\frac{1}{25}(W - 300)\right)$

$= \frac{1}{625}(W - 300)$

1pt - $\frac{d^2W}{dt^2}$

1pt - answer w/ reason

$\left.\frac{d^2W}{dt^2}\right|_{t=\frac{1}{4}} > 0$, $\therefore$ part (a) is an underestimate

$\frac{d^2W}{dt^2} > 0$ under

$\frac{d^2W}{dt^2} < 0$ over

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

y with y's
x w/ x's

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $\boxed{W(0) = 1400.}$ ← initial condition

$$\frac{dW}{dt} = \frac{1}{25}(W-300)$$

$$dW = \frac{1}{25}(W-300)\,dt$$

$$\int \frac{1}{W-300}\,dW = \int \frac{1}{25}\,dt$$

1pt - separate
1pt - antiderive
1pt - "+c"
1pt - initial condition

$u = W-300$

$\frac{du}{dW} = 1$

$du = dW$

$$\int \frac{1}{u} \cdot du = \frac{1}{25}t + C$$

$$\ln|u| = \frac{1}{25}t + C$$

$$\ln|W-300| = \frac{1}{25}t + C$$

$$\ln|1400-300| = \frac{1}{25}(0) + C$$

$$\ln 1100 = C$$

$$\ln|W-300| = \frac{1}{25}t + \ln 1100$$

$$|W-300| = e^{\frac{1}{25}t + \ln 1100}$$

$$W-300 = \pm e^{\frac{1}{25}t + \ln 1100}$$

$$W-300 = e^{\frac{1}{25}t + \ln 1100}$$

(keep "+" b/c $W-300$ is pos when $W=1400$)

$$W = e^{\frac{1}{25}t + \ln 1100} + 300$$ ← ok to stop here

1pt - solve for W

OR

$$W = e^{\frac{1}{25}t} e^{\ln 1100} + 300$$

$$W = 1100e^{\frac{1}{25}t} + 300$$

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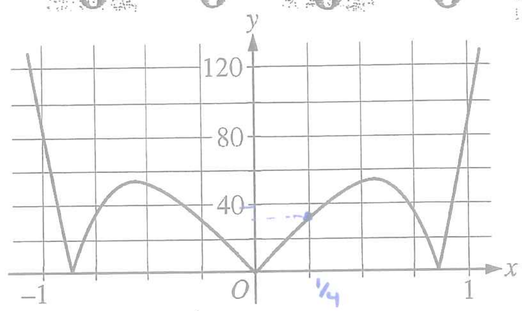

Graph of $y = \left|f^{(5)}(x)\right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ 1pt - sin x series

$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$ 2pt - sin(x²) series

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ 1pt - cos x series

$f(x) = \sin(x^2) + \cos x$

$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ 2pt - f(x) series

$= 1 + \frac{1}{2}x^2 + \frac{x^4}{4!} \underbrace{- \frac{x^6}{3!} - \frac{x^6}{6!}}\ + \cdots$

$- \frac{x^6}{6} - \frac{x^6}{6\underbrace{(5\cdot4\cdot3\cdot2\cdot1)}_{120}}$

$\left(-\frac{121x^6}{720}\right)$

Continue problem 6 on page 15.