NO CALCULATOR ALLOWED

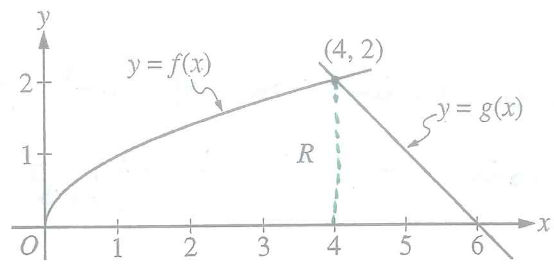
CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



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3. The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

$$\text{Area of } R = \int_0^4 \sqrt{x}\,dx + \int_4^6 (6-x)\,dx$$

$$= \frac{2}{3}x^{3/2}\Big|_0^4 + \left(6x - \frac{1}{2}x^2\right)\Big|_4^6$$

$$= \frac{2}{3}(4)^{3/2} + \left(6(6) - \frac{1}{2}(6)^2 - \left(6\cdot 4 - \frac{1}{2}(4)^2\right)\right)$$

ok to stop here.

$$= \frac{16}{3} + 36 - 18 - 24 + 8$$

$$= \frac{16}{3} + 2 = \boxed{\frac{22}{3}}$$

1pt - integral
1pt - antiderivative
1pt - answer

OR

$$\text{Area of } R = \int_0^2 (6 - y - y^2)\,dy$$

$$= \left(6y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^2$$

$$= 12 - 2 - \frac{8}{3}$$

$$= 22/3$$

$y = 6 - x$
$y + x = 6$
$x = 6 - y$

$y = \sqrt{x}$
$y^2 = x$

Continue problem 3 on page 9.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

walk in y's

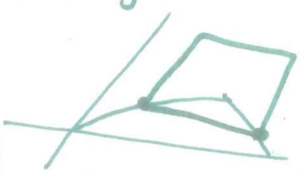
work in y's

$y = 6 - x \qquad y = \sqrt{x}$

$x = 6 - y \qquad y^2 = x$

$A = bh$

$A = (6 - y - y^2)(2y)$



base $= (6 - y - y^2)$

Volume $= \int_0^2 (6 - y - y^2)(2y)\, dy$

2pts - integrand
1pt - answer

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.

$\to (x, y)$

$f'(x) \quad =$

need slope of g and then opp. reciprocal.

$g'(x) = -1$

m $\perp$ to g is 1

~~POF~~

$f'(x) = \frac{1}{2} x^{-1/2}$

$1 = \frac{1}{2} x^{-1/2}$

$2 = x^{-1/2} \to \frac{2}{1} = \frac{1}{x^{1/2}}$

$\frac{1}{2} = x^{1/2}$

$\frac{1}{4} = x$

$f(\frac{1}{4}) = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Point P is $(\frac{1}{4}, \frac{1}{2})$

1pt - $f'(x)$
1pt - equation
1pt - answer

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GO ON TO THE NEXT PAGE.

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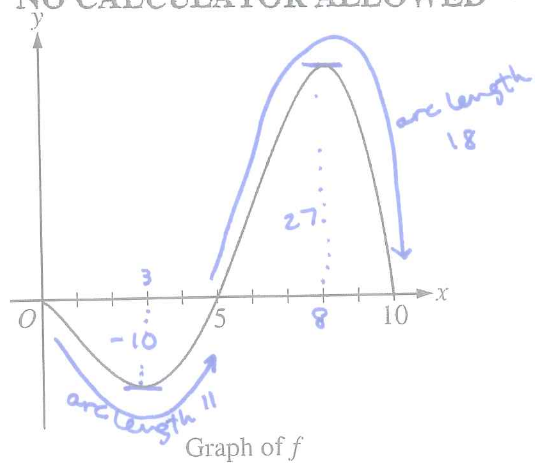

Graph of f

4. The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.

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AB

(a) Find the average value of f on the interval $0 \le x \le 5$.

$\frac{1}{b-a}\int_a^b f(t)\,dt$

avg value of f $= \frac{1}{5-0}\int_0^5 f(t)\,dt$

$= \frac{1}{5}(-10)$

$= -2$

1pt - answer

AB

(b) Evaluate $\int_0^{10} (3f(x) + 2)\,dx$. Show the computations that lead to your answer.

$3\int_0^{10} f(x)\,dx + \int_0^{10} 2\,dx$

$3(-10 + 27) + 2x\Big|_0^{10}$

$3(17) + 20$

$51 + 20$

71

2pts - answer

Continue problem 4 on page 11.

AB

(c) Let $g(x) = \int_5^x f(t)\,dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

$g'' > 0$ $\quad$ $g' < 0$

$g(x) = \int_5^x f(t)\,dt$

$g'(x) = \frac{d}{dx}\int_5^x f(t)\,dt$

$g'(x) = f(x)$

g': $-$ on $(0,5)$, $+$ on $(5,10)$; marks at 0, 5, 10

$g' < 0$ on $(0,5)$ b/c $f(x) < 0$ on $(0,5)$

$g''(x) = f'(x)$

g'': $-$, $+$, $-$; marks at 3, 8

f inc

$g''(x) > 0$ on $(3,8)$ b/c $f'(x) > 0$ on $(3,8)$

g conc up and dec on $(3,5)$ b/c $g'' > 0$ and $g' < 0$ on $(3,5)$

1pt – $g'(x) = f(x)$

1pt – analysis

1pt – answer w/ reason

BC

(d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

$\int \sqrt{1+(f'(x))^2}$

Arc Length of $h(x)$ $= \int_0^{20} \sqrt{1+(h'(x))^2}\,dx$

$= \int_0^{20} \sqrt{1+\left(f'\left(\frac{x}{2}\right)\right)^2}\,dx$

$u = \frac{x}{2}$ $\quad$ $u(0) = 0$

$\frac{du}{dx} = \frac{1}{2}$ $\quad$ $u(20) = 10$

$2\,du = dx$

$= \int_0^{10} \sqrt{1+(f'(u))^2} \cdot 2\,du$

$= 2\int_0^{10} \sqrt{1+(f'(u))^2}\,du$

$= 2(11+18)$

$= 58$

1pt – integral

1pt – substitution

1pt – answer

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NO CALCULATOR ALLOWED

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

5. Ben rides a unicycle back and forth along a straight east-west track. The $\boxed{\text{twice-differentiable}}$ function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

↳ $v'(t)$

$$a(5) = \frac{v(10) - v(0)}{10 - 0} \quad \frac{m/sec}{sec} \ldots$$

$$= \frac{2.3 - 2.0}{10}$$

$$= \frac{.3}{10}\ m/sec^2 \quad \text{OR} \quad .03\ m/sec^2$$

1 pt - answer

1 pt - units in (a) or (b)

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\, dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\, dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

$$\int_0^{60} |v(t)|\, dt = 10(2) + 30(2.3) + 20(2.5)$$

$$= 20 + 69 + 50$$

$$= 139 \text{ meters}$$

1 pt - left approx.

$\int_0^{60} |v(t)|\, dt$ is the total distance Ben traveled on his unicycle in meters from $t=0$ to $t=60$ sec.

1 pt - meaning

Continue problem 5 on page 13.

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IVT, MVT

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

MVT $B'(t) = \frac{B(b)-B(a)}{b-a}$

$B(t)$ is cont + diff'able b/c B is twice-diff'able

or $v(t)$

$B'(t) = \frac{B(60)-B(40)}{60-40}$

$= \frac{49-9}{20}$

$= 2$

1pt - difference quotient

Since $B'(t)=2$ on $(40,60)$, there must be a time t when $v(t)=2$ on $(40,60)$

1pt - conclusion w/ reason

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(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

$L'(40) = ?$

$[L(t)]^2 = 12^2 + (B(t))^2$

$2(L(t)) \cdot L'(t) = 2B(t) \cdot B'(t)$

1pt - derivatives

1pt - use $B'(t) = v(t)$

$2L(40) \cdot L'(40) = 2B(40) \cdot B'(40)$

$(L(40))^2 = 12^2 + (B(40))^2$

$[L(40)]^2 = 12^2 + 9^2$

$2(15) \cdot L'(40) = 2(9) \cdot 2.5$

$L(40) = \sqrt{144+81}$

$= \sqrt{225}$

$= 15$

$L'(40) = \frac{\cancel{2} \cdot 9 \cdot 2.5}{\cancel{2} \cdot 15}$

$\cancel{18}$ / 4.5 / $\cancel{22.5}$ / 16

$L'(40) = \frac{3(\frac{5}{2})}{5} = 3(\frac{5}{2}) \div 5 = \frac{3}{1}(\frac{5}{2}) \cdot \cancel{\frac{1}{5}}$

$= \frac{3}{2}$ m/sec

1pt - answer

6. Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

$\ln(1+x^3) = x^3 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \dots + (-1)^{n+1}\frac{(x^3)^n}{n} + \dots$

OR

$= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \dots + \frac{(-1)^{n+1}x^{3n}}{n} + \dots$

1pt – 1st four terms

1pt – general term

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

$|x-a| < R$

$|x| < 1 \qquad -1 < x < 1$

check endpts:

$x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{3n}}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^{4n}}{n}$

$= \sum_{n=1}^{\infty} \frac{-1}{n}$

diverges, p-test w/ $p=1$

$x = 1 \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{3n}}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

* AST *

① $\frac{1}{n} > 0$ for $(1, \infty)$

② $\frac{1}{n} > \frac{1}{n+1}$ b/c $n+1 > n$

③ $\lim_{n\to\infty} \frac{1}{n} = 0$

so converges

2pts – answer w/ reason

$\therefore$, interval of convergence is : $-1 < x \le 1$

Continue problem 6 on page 15.