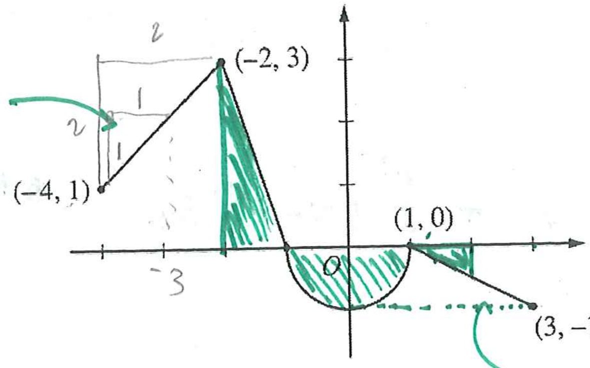
eq.
$y-1=1(x+4)$
$y=x+5$

$y(-3)=2$



Graph of f

eq. $y-0=-\frac{1}{2}(x-1)$
$y=-\frac{1}{2}x+\frac{1}{2}$

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$g(2) = \int_1^2 f(t)\,dt$

$= \frac{1}{2}(1)(-\frac{1}{2})$

$= -\frac{1}{4}$

$y(2) = -\frac{1}{2}(2)+\frac{1}{2}$
$= -\frac{1}{2}$

1pt - g(2)

$g(-2) = \int_1^{-2} f(t)\,dt$

$= -\int_{-2}^{1} f(t)\,dt = -\left[\left(-\frac{\pi(1)^2}{2}\right)+\frac{1}{2}(1)(3)\right] = \frac{\pi}{2} - \frac{3}{2}$

1pt - g(-2)

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = \frac{d}{dx}\int_1^x f(t)\,dt$

$= f(x)$

$g'(-3) = f(-3)$

$= 2$

$g''(x) = f'(x)$

$g''(-3) = f'(-3)$

$= 1$

1pt - g'(-3)

1pt - g''(-3)

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Continue problem 3 on page 15.

$g'=0$

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x)=0 \rightarrow g$ has hor. tan. line

1pt – $g'(x)=0$

$g'(x)=f(x)=0$

@ $x=-1$, $x=1$

g' + | − | −
-1 1

1pt – $x=-1$ and $x=1$

g has rel. max @ $x=-1$ b/c g' changes from pos to neg.

@ $x=1$
g has neither rel. max nor min b/c g' doesn't change signs.

1pt – answers w/ reasons

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Do not write beyond this border.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

↳ g'' changes signs

$g''(x)=f'(x)$

$g''=0$ @ $x=0$ $\quad$ g'' DNE @ $x=-2,-1,1$

g'' + | − | − | + | −
-2 -1 0 1

g has pt. of inf @ $x=-2$, $x=1$, and $x=0$ b/c g'' changes signs.

1pt – answer

1pt – reason

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

mdpt mdpt

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

4. The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

AB

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

$\to y - y_1 = m(x - x_1)$

$f(1) = 15$

$f'(1) = 8$

$y - 15 = 8(x - 1)$

$y - 15 = 8(1.4 - 1)$

$y - 15 = 8(.4)$

$y - 15 = 3.2$

$y = 18.2$

$f(1.4) = 18.2$

1pt - tangent line

1pt - approx of $f(1.4)$

AB

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

$\int_1^{1.4} f'(x)\,dx = .2(10) + .2(13)$

$= 2.0 + 2.6$

$= 4.6$

1pt - mdpt sum

$\int_1^{1.4} f'(x)\,dx = f(x)\Big|_1^{1.4}$

1pt - FTC

$\int_1^{1.4} f'(x)\,dx = f(1.4) - f(1)$

$4.6 = f(1.4) - 15$

$19.6 \approx f(1.4)$

1pt - answer

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

BC

$y = \frac{dy}{dx}(\Delta x) + y$ $\quad \Delta x = .2$

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

	Δx	$\frac{dy}{dx}$	$\frac{dy}{dx}(\Delta x)$	$\frac{dy}{dx}(\Delta x) + y$
$(1, 15)$	$.2$	8	$8(.2) = 1.6$	$1.6 + 15 = 16.6$
$(1.2, 16.6)$	$.2$	12	$12(.2) = 2.4$	$2.4 + 16.6 = 19$
$(1.4, 19)$				

1pt - Euler's w/2 steps

$f(1.4) \approx 19$

1pt - answer

BC

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!}$$

$$= 15 + 8(x-1) + \frac{20}{2}(x-1)^2$$

1pt - Taylor polynomial

$$f(1.4) \approx 15 + 8(1.4-1) + 10(1.4-1)^2$$

$$\approx 15 + 8(.4) + 10(.4)^2$$

$$\approx 15 + 3.2 + 10(.16)$$

$$\approx 15 + 3.2 + 1.6$$

$$\approx 19.8$$

1pt - approximation

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5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

↳ rate

$$\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100-40) = \frac{60}{5} = 12 \text{ grams/day}$$

1pt - uses $\frac{dB}{dt}$

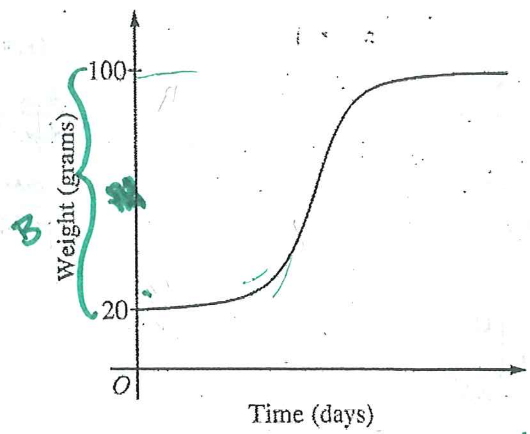
$$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100-70) = \frac{30}{5} = 6 \text{ grams/day}$$

1pt - answer w/ reason

Bird gaining weight faster when weighs 40 grams

b/c $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



graph shows B btn 20 + 100

$\frac{d^2B}{dt^2}$ 20 30 100

Since $\frac{d^2B}{dt^2} < 0$ when B is (20, 100), B should be concave down on (20, 100), but the graph is concave up for some of the interval.

1pt - $\frac{d^2B}{dt^2}$

$$\frac{d^2B}{dt^2} = \frac{1}{5}(-1)\frac{dB}{dt}$$

$$= -\frac{1}{5}\left(\frac{1}{5}(100-B)\right)$$

$$0 = -\frac{1}{25}(100-B) \rightarrow B = 100$$

1pt - explanation

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Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $\boxed{B(0) = 20}$.

$\frac{dB}{dt} = \frac{1}{5}(100-B)$

$dB = \frac{\frac{1}{5}(100-B)\,dt}{(100-B)}$

1pt-separate
1pt-antiderive
1pt-"+c"
$u = 100-B$
$\frac{du}{dB} = -1$
$du = -dB$

$\int \frac{1}{100-B}\,dB = \int \frac{1}{5}\,dt$

$\int \frac{1}{u}\cdot -du = \frac{1}{5}t + c$

$-\ln|u| = \frac{1}{5}t + c$

$-\ln|100-B| = \frac{1}{5}t + c$

1pt initial condition

$-\ln|100-20| = \frac{1}{5}(0) + c$

$-\ln 80 = c$

$-\ln|100-B| = \frac{1}{5}t - \ln 80$

$\ln|100-B| = -\frac{1}{5}t + \ln 80$

$|100-B| = e^{-\frac{1}{5}t + \ln 80}$

$100-B = e^{-\frac{1}{5}t + \ln 80}$

$-B = e^{-\frac{1}{5}t + \ln 80} - 100$

1pt-solves for B

$\boxed{B = 100 - e^{-\frac{1}{5}t + \ln 80}}$

or $B = 100 - 80e^{-\frac{1}{5}t}$

OR

$\frac{dB}{20-\frac{1}{5}B} = \frac{(20-\frac{1}{5}B)\,dt}{20-\frac{1}{5}B}$

$\int \frac{1}{20-\frac{1}{5}B}\,dB = \int dt$

$\int \frac{1}{u}\cdot\frac{du}{-\frac{1}{5}} = t + C$

$u = 20 - \frac{1}{5}B$

$\frac{du}{dB} = -\frac{1}{5}$

$\frac{du}{-\frac{1}{5}} = dB$

$-5\int \frac{1}{u}\,du = t + C$

$-5\ln|u| = t + C$

$-5\ln|20-\frac{1}{5}B| = t + C$

$-5\ln|20-\frac{1}{5}20| = C$

$-5\ln 16 = C$

$-5\ln|20-\frac{1}{5}B| = t - 5\ln 16$

$\ln|20-\frac{1}{5}B| = \frac{t-5\ln 16}{-5}$

$|20-\frac{1}{5}B| = e^{\frac{t-5\ln 16}{-5}}$

$20-\frac{1}{5}B = e^{\frac{t-5\ln 16}{-5}}$

$\frac{-\frac{1}{5}B = e^{\frac{t-5\ln 16}{-5}} - 20}{}$

$B = -5\left(e^{\frac{t-5\ln 16}{-5}} - 20\right)$

GO ON TO THE NEXT PAGE.

6. The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+3}=\frac{x}{3}-\frac{x^3}{5}+\frac{x^5}{7}-\cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty}\frac{(-1)^{n+1}x^{2(n+1)+1}}{2(n+1)+3}\cdot\frac{2n+3}{(-1)^n x^{2n+1}}$$

1pt - set up ratio

$$=\lim_{n\to\infty}(-1)x^2\left(\frac{2n+3}{2n+5}\right)$$

1pt - compute ratio

$$=-x^2$$

$$|-x^2|<1$$

$$|x^2|<1$$

$$|x|<1$$

$$-1<x<1$$

1pt - interval

check endpts:

$x=-1$ $\quad\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+3}$

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{2n+3}$

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}$

$x=1$ $\quad\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+3}$

$=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}$

same series

So also converges by alt. series test

1pt - consider endpts

AST

① $\frac{1}{2n+3}>0$ for $(0,\infty)$

② $\frac{1}{2n+3}>\frac{1}{2(n+1)+3}$

b/c $2n+5>2n+3$

③ $\lim_{n\to\infty}\frac{1}{2n+3}=0$

So converges by alternating series test

1pt - analysis and correct interval of convergence

$\therefore$, interval of convergence is : $-1\le x\le 1$

Continue problem 6 on page 21.

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(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

terms after 1st 2 are smaller than 3rd term

show

$$\left|g\left(\tfrac{1}{2}\right) - \frac{17}{120}\right| < \frac{1}{200}$$

$$\left|g\left(\tfrac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7}$$

$$\leq \frac{\frac{1}{32}}{7}$$

$$< \frac{1}{224}$$

so, $\left|g\left(\tfrac{1}{2}\right) - \frac{17}{120}\right| < \frac{1}{200}$

1 pt - use 3rd term for error bound

1pt - error bound

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$$g(x) = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+3} + \cdots$$

$$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + \frac{(-1)^n \cdot (2n+1)\, x^{2n}}{2n+3} + \cdots$$

1pt - 3 terms

1pt - general term

GO ON TO THE NEXT PAGE.