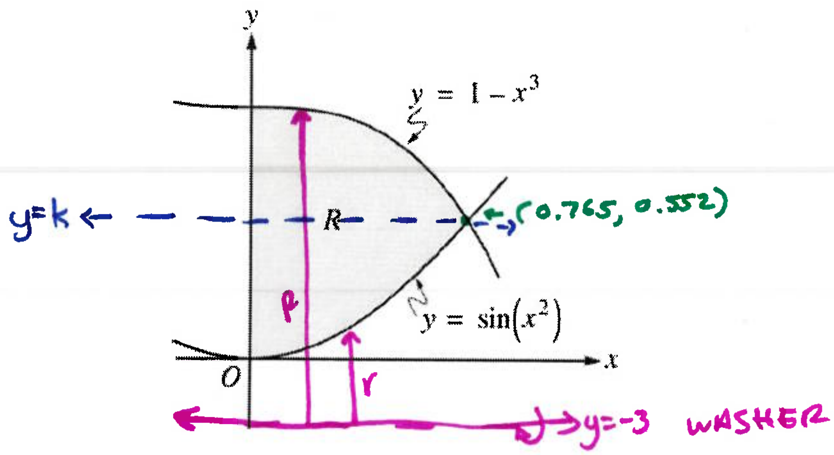

1. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $\underbrace{y = 1 - x^3}_{f1}$ and $\underbrace{y = \sin(x^2)}_{f2}$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area of } R = \int_0^{0.765} \underbrace{\left(1 - x^3 - \sin(x^2)\right)}_{\text{1 pt: integrand}} dx$$

$$= 0.534$$

1 pt: answer

1 pt: Limits correct in (a), (b), or (c)

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Continue problem 1 on page 5.

y-value of intersection pt.

(b) A horizontal line, $y = k$, is drawn through the point where the graphs of $y = 1 - x^3$ and $y = \sin(x^2)$ intersect. Find k and determine whether this line divides R into two regions of equal area. Show the work that leads to your conclusion. $k = 0.552$

Area above $y=k$ $= \int_0^{0.765} (1 - x^3 - k)\,dx$

$= 0.257$

1pt: integral(s) w/ k-value

1pt: value(s) of integral(s)

Area below $y=k$ $= \int_0^{0.765} (k - \sin(x^2))\,dx$

$= 0.277$

The two regions are not equal areas.

1pt: conclusion tied to part (a) or comparison of the two integrals

---

(c) Find the volume of the solid generated when R is revolved about the line $y = -3$.

$\text{Volume} = \pi \int \left((\text{outside radius})^2 - (\text{inside radius})^2 \right) dx$ ☺

$\text{Volume} = \pi \int_0^{0.765} \left[\underbrace{(1 - x^3 - -3)^2 - (\sin(x^2) - -3)^2}_{\text{2 pts: integrand}} \right] dx$

$= 11.841$

1pt: answer

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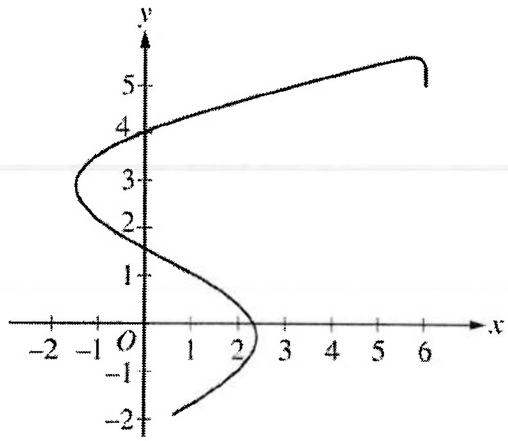

2. A planetary rover travels on a flat surface. The path of the rover for the time interval $0 \le t \le 2$ hours is shown in the rectangular coordinate system above. The rover starts at the point with coordinates $(6, 5)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position of the rover change at rates given by

$x(0)=6$ $y(0)=5$

Velocity
$$x'(t) = -12\sin\left(2t^2\right)$$
$$y'(t) = 10\cos\left(1 + \sqrt{t}\right),$$

where $x(t)$ and $y(t)$ are measured in meters and t is measured in hours.

(a) Find the acceleration vector of the rover at time $t = 1$. Find the speed of the rover at time $t = 1$.

↳ $a(t) = v'(t) = \langle x'', y'' \rangle$

↳ $|v(t)| = \sqrt{(x')^2 + (y')^2}$

$a(1) = \langle x''(1), y''(1) \rangle$

$= \langle 19.975, -4.546 \rangle$

1pt: acceleration vector

speed @ $t=1$ $= \sqrt{(x'(1))^2 + (y'(1))^2}$

$= 11.678$

1pt: speed

Continue problem 2 on page 7.

(b) Find the total distance that the rover travels over the time interval $0 \le t \le 1$.

↳ $\int |v(t)|\, dt$

Total distance $= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ — 2pts: integral

$= 6.704$ — 1pt: answer

(c) Find the y-coordinate of the position of the rover at time $t = 1$.

↳ $\int y'(t)$

$y(1) = y(0) + \int_0^1 y'(t)\, dt$

$= 5 + \int_0^1 y'(t)\, dt$ — 1pt: integral

1pt: answer

$= 4.057$

(d) The rover receives a signal at each point where the line tangent to its path has slope $\frac{1}{2}$. At what times t, for $0 \le t \le 2$, does the rover receive a signal?

when $\frac{dy}{dx} = \frac{1}{2}$?

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{1}{2} = \frac{10\cos(1+\sqrt{t})}{-12\sin(2t^2)}$ — 1pt: equation

1pt: answer

$t = 1.072$

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$\Delta t = 10$ $\Delta t = 12$ $\Delta t = 8$

t (days)	0	10	22	30
$W'(t)$ (GL per day)	0.6	0.7	1.0	0.5

L L L

3. The twice-differentiable function W models the volume of water in a reservoir at time t, where $W(t)$ is measured in gigaliters (GL) and t is measured in days. The table above gives values of $W'(t)$ sampled at various times during the time interval $0 \le t \le 30$ days. At time $t = 30$, the reservoir contains 125 gigaliters of water.

$y - y_1 = m(x - x_1)$

(a) Use the tangent line approximation to W at time $t = 30$ to predict the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 32$. Show the computations that lead to your answer.

$W - W_1 = W'(t)(t - t_1)$

$W(32) - W(30) = W'(30)(32 - 30)$

$W(32) - 125 = 0.5(32 - 30)$

$W(32) = 0.5(32-30) + 125$

$= 126$ gigaliters

1pt: answer

(b) Use a left Riemann sum, with the three subintervals indicated by the data in the table, to approximate $\int_0^{30} W'(t)\,dt$. Use this approximation to estimate the volume of water $W(t)$, in gigaliters, in the reservoir at time $t = 0$. Show the computations that lead to your answer.

need $W(0)$

$\int_0^{30} W'(t)\,dt \approx 10(0.6) + 12(0.7) + 8(1.0)$

1pt: left sum

$= 22.4$ gigaliters

1pt: approx

$\int_0^{30} W'(t)\,dt = W(t)\Big|_0^{30}$

$\int_0^{30} W'(t)\,dt = W(30) - W(0)$ ← isolate $W(0)$ ... ☺

$\therefore,\ W(0) = W(30) - \int_0^{30} W'(t)\,dt$

$= 125 - 22.4$

1pt: answer

$= 102.6$ gigaliters

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) Explain why there must be at least one time t, other than $t = 10$, such that $W'(t) = 0.7$ GL/day.

W' is diff'able → ∴, W' is continuous

$\left.\begin{array}{l} W'(22) = 1.0 \\ W'(30) = 0.5 \end{array}\right\}$ 0.7 is btn 1.0 & 0.5

By IVT, there must be at least one time t, other than $t = 10$, on [22, 30] s.t. $W'(t) = 0.7$ b/c $W'(30) < 0.7 < W'(22)$. and W' is cont.

2pts : explanation

(d) The equation $A = 0.3W^{2/3}$ gives the relationship between the area A, in square kilometers, of the surface of the reservoir, and the volume of water $W(t)$, in gigaliters, in the reservoir. Find the instantaneous rate of change of A, in square kilometers per day, with respect to t when $t = 30$ days.

derivative

$\left.\frac{dA}{dt}\right|_{t=30} = ?$

$A = 0.3W^{2/3}$

$\frac{dA}{dt} = 0.3(\frac{2}{3}W^{-1/3})\frac{dW}{dt}$

W(30) initial condition

$\left.\frac{dW}{dt}\right|_{t=30} = W'(30) = 0.5$ see table

$\left.\frac{dA}{dt}\right|_{t=30} = 0.3(\frac{2}{3}\cdot(125)^{-1/3})\cdot(0.5)$

$= 0.02$

2pts : $\frac{dA}{dt}$

1pt : answer

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NO CALCULATOR ALLOWED

4. Consider the function f given by $f(x) = xe^{-x^2}$ for all real numbers x.

(a) At what value of x does $f(x)$ attain its absolute maximum? Justify your answer.

↳ crit # ~~or endpt~~... no end pt, so justify why rel. max is abs. max

$f'(x) = e^{-x^2}(1) + x(-2xe^{-x^2})$

$= e^{-x^2}(1 - 2x^2)$

$\cancel{0 = e^{-x^2}}$ $\quad 1 - 2x^2 = 0$

$1 = 2x^2$

$\frac{1}{2} = x^2$

$\pm\sqrt{\frac{1}{2}} = x$

f': $-$ $\quad$ $+$ $\quad$ $-$

$-10 \quad -\sqrt{\frac{1}{2}} \quad 0 \quad \sqrt{\frac{1}{2}} \quad 10$

2 pts: $f'(x)$

1 pt: solves $f'(x) = 0$

f has rel. max @ $x = \sqrt{\frac{1}{2}}$ b/c f' changes from pos to neg @ $x = \sqrt{\frac{1}{2}}$

f is neg on $(-\infty, 0)$ and f is pos on $(0, \infty)$. $\therefore$, the abs max can only occur on $(0, \infty)$

$\boxed{\therefore, \text{abs max @ } x = \sqrt{\tfrac{1}{2}}}$

1 pt: answer

1 pt: reason

(b) Find an antiderivative of f. +C

$\int xe^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^u + C$

$= \boxed{-\frac{1}{2}e^{-x^2} + C}$

1 pt: antiderivative

Continue problem 4 on page 17.

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NO CALCULATOR ALLOWED

(c) Find the value of $\int_0^\infty x f(x)\,dx$, given the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

$\int_0^\infty x(xe^{-x^2})\,dx = \int_0^\infty x^2 e^{-x^2}\,dx$

$= \lim_{a\to\infty} \int_0^a x^2 e^{-x^2}\,dx$

$= \lim_{a\to\infty} \left(-\frac{1}{2}xe^{-x^2}\Big|_0^a - \int_0^a -\frac{1}{2}e^{-x^2}\,dx\right)$

$= \lim_{a\to\infty} \left(-\frac{1}{2}ae^{-a^2} + \int_0^a \frac{1}{2}e^{-x^2}\,dx\right)$

$= 0 + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$

$= \frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right)$

$= \boxed{\frac{\sqrt{\pi}}{4}}$

u	dv
x	xe^{-x^2}
1	$-\frac{1}{2}e^{-x^2}$
0	

$uv - \int v\,du$

$-\frac{1}{2}xe^{-x^2} - \int -\frac{1}{2}e^{-x^2}\,dx$

2pts: integration by parts

1pt: answer

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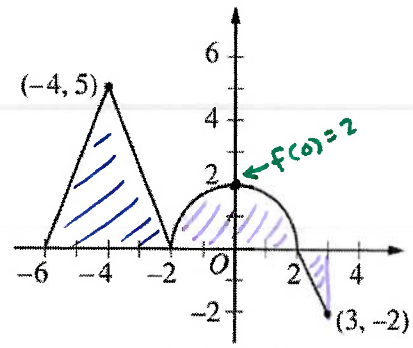
NO CALCULATOR ALLOWED



Graph of f

5. The graph of the continuous function f, consisting of three line segments and a semicircle, is shown above. Let g be the function given by $g(x) = \int_{-2}^{x} f(t)\,dt$.

(a) Find $g(-6)$ and $g(3)$.

$$g(-6) = \int_{-2}^{-6} f(t)\,dt$$

$$= -\int_{-6}^{-2} f(t)\,dt$$

$$= -\left(\tfrac{1}{2}(4)(5)\right)$$

$$\boxed{g(-6) = -10}$$

Area △

$$g(3) = \int_{-2}^{3} f(t)\,dt$$

ok to stop here for AP

$$= \tfrac{1}{2}\pi(2)^2 + \tfrac{1}{2}(1)(-2)$$

$$\boxed{g(3) = 2\pi - 1}$$

Area semicircle and △

1pt: $g(-6)$

1pt: $g(3)$

(b) Find $g'(0)$.

$$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$$

$$g'(x) = f(x)$$

$$g'(0) = f(0)$$

$$\boxed{g'(0) = 2}$$

1pt: $g'(0)$

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a horizontal tangent. Determine whether g has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

$\to g'(x) = 0$

$\hookrightarrow g'$ pos to neg $\hookrightarrow g'$ neg to pos

$g'(x) = f(x) = 0$

@ $x = -6, -2, 2$

↑ not in interval $(-6,3)$ so only $x = -2$, and $x = 2$

1 pt: horizontal tangents @ $x = -2$ and $x = 2$

g does not have rel. max or min @ $x = -2$ b/c g' does not change signs @ $x = -2$

2 pts: answers w/ reasons

g has rel. max @ $x = 2$ b/c g' changes from pos to neg @ $x = 2$.

(d) Find all values of x on the open interval $-6 < x < 3$ for which the graph of g has a point of inflection. Explain your reasoning.

$g'' = 0$ or DNE p.i.p's

g'' changes signs

$g'(x) = f(x)$

$g''(x) = f'(x) = 0$ or DNE

@ $x = -4$, $x = -2$, $x = 0$, $x = 2$ ← possible inf. pts.

2 pts: values of x

g has inf pts @ $x = -4$, $x = -2$, and $x = 0$ b/c g'' changes signs at these x-values.

1 pt: reason

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NO CALCULATOR ALLOWED

6. The function f satisfies the equation

$$f'(x) = f(x) + x + 1$$

x_1 y_1

and $f(0) = 2$. The Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

(a) Write an equation for the line tangent to the curve $y = f(x)$ at the point where $x = 0$.

$\hookrightarrow y - y_1 = m(x - x_1)$

$f'(0) = f(0) + 0 + 1$
$= 2 + 0 + 1$
$= 3$

$\boxed{y - 2 = 3\,(x - 0)}$

1pt: tangent line equation

(b) Find $f''(0)$ and find the second-degree Taylor polynomial for f about $x = 0$.

$f'(x) = f(x) + x + 1$
$f''(x) = f'(x) + 1$
$= f(x) + x + 1 + 1$
$= f(x) + x + 2$

$f''(0) = f(0) + 0 + 2$

$\boxed{f''(0) = 4}$

1pt: $f''(0)$

$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2$

$\approx 2 + 3x + \frac{4}{2!}x^2$

$\boxed{f(x) \approx 2 + 3x + 2x^2}$

OR

$T_2(x) = 2 + 3x + 2x^2$

← 1pt: 2nd degree Taylor poly.

Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(c) Find the fourth-degree Taylor polynomial for f about $x = 0$.

$f'''(x) = f'(x) + 1$
$f'''(x) = f''(x)$
$f'''(0) = f''(0)$
$= 4$

$f^{IV}(x) = f'''(x)$
$f^{IV}(0) = f'''(0)$
$= 4$

1pt: $f'''(0)$ and $f^{(4)}(0)$

$T_4(x) = 2 + 3x + 2x^2 + \frac{4}{3!}x^3 + \frac{4}{4!}x^4$

$\boxed{T_4(x) = 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4}$

1pt: 4th degree Taylor poly.

(d) Find $f^{(n)}(0)$, the nth derivative of f at $x = 0$, for $n \geq 2$. Use the Taylor series for f about $x = 0$ and the Taylor series for e^x about $x = 0$ to find a polynomial expression for $f(x) - 4e^x$.

$f'''(0) = 4$
$f^{IV}(0) = 4$
$\vdots$
$f^{(n)}(0) = 4$ for $n \geq 2$

1pt: $f^{(n)}(0)$

$f(x) = 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \cdots$

$-\ 4e^x = 4(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots)$

$\boxed{f(x) - 4e^x = -2 - x}$

all other terms cancel out ... ☺

1pt: Taylor series for f

1pt: Taylor series for e^x

1pt: polynomial expression

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