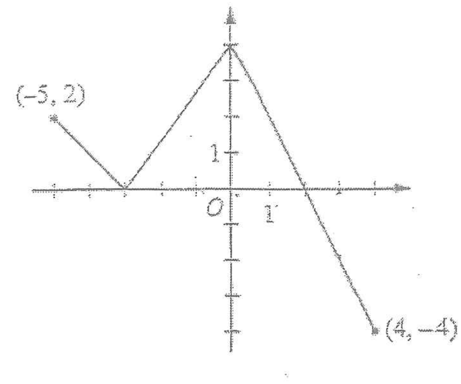

Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$g(x) = \int_{-3}^{3} f(t)\,dt$

$= \frac{1}{2}(5)(4) + \frac{1}{2}(1)(-2)$

$= 10 - 1$

$= 9$

1pt – answer

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$g' > 0$ $\quad g'' < 0$

$g'(x) = \frac{d}{dx} \int_{-3}^{x} f(t)\,dt$

$g'(x) = f(x)$ $\quad g'(x)$: $+$ | -3 | $+$ | 2 | $-$ $\rightarrow$ $g' > 0$ on $(-5, 2)$

$g'(x) = f(x) = 0 \rightarrow$

$g''(x) = f'(x)$ $\quad g''$: $-$ | -3 | $+$ | 0 | $-$ $\rightarrow$ $g'' < 0$ on $(-5, -3) \cup (0, 4)$

$f'(x) = 0$ or DNE $\rightarrow$

g inc & conc. down on $(-5, -3) \cup (0, 2)$ b/c $g' > 0$ & $g'' < 0$ on $(-5, -3) \cup (0, 2)$

1pt – answer
1pt – reason

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Continue problem 3 on page 15.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

← quotient rule

$$h'(x) = \frac{5x(g'(x)) - g(x)\cdot 5}{(5x)^2}$$

2pts – h'(x)

$$h'(3) = \frac{5\cdot 3\, g'(3) - g(3)\cdot 5}{(5\cdot 3)^2}$$

$g'(x) = f(x)$
$g'(3) = f(3)$
$g'(3) = -2$

:)

$$= \frac{15(-2) - 9\cdot 5}{(15)^2}$$

$$= -\frac{1}{3}$$

1pt – answer

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

chain!

↳ $p'(x)$ @ $x = -1$

$$p'(x) = f'(x^2 - x)\cdot(2x-1)$$

2pts – p'(x)

$$p(-1) = f'((-1)^2 - -1)\cdot(2(-1)-1)$$

$$= f'(2)\cdot(-3)$$

$$= -2(-3)$$

$$= 6$$

1pt – answer

:)

$f'(2) = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-8}{4}$

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NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

avg acceleration of Train A $= \dfrac{V_A(8) - V_A(2)}{8-2}$ $\dfrac{\text{meters/min}}{\text{min}}$ ... ☹

$= \dfrac{-120-100}{6}$

$= \dfrac{-220}{6}$

$= -\dfrac{110}{3}$ meters/min/min

or $-\dfrac{110}{3}$ meters/min^2

1pt - avg. acceleration

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

MVT? IVT?

Train A velocity is diff'able, ∴, velocity is cont.

Since $V_A(5) > -100$ and $V_A(8) < -100$,

← 1pt - compare $V_A(5)$ & $V_A(8)$ to −100

then ∃ some time t on (5, 8) s.t. train A's velocity is −100 m/min.

(IVT)

1pt - conclusion, using IVT

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

initial time   initial position

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$. → trap $= \frac{1}{2}(b_1+b_2)h$

$$x_A(12) = 300 + \int_2^{12} v_A(t)\,dt$$

1pt – position

$$= 300 + \left[\tfrac{1}{2}(100+40)(3) + \tfrac{1}{2}(40+-120)(3) + \tfrac{1}{2}(-120+-150)(4)\right]$$

← 1pt – trapezoid sum

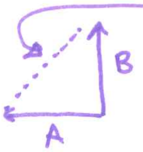
$$= 300 + \tfrac{1}{2}(140\cdot 3 + -80\cdot 3 + -270\cdot 4)$$

$$= -150 \text{ meters}$$

Train is 150 meters west of station

← 1pt – position @ $t=12$

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$A^2 + B^2 = C^2$

@ $t = 2$,
$A = 300$ (given in part c)
$B = 400$

$\frac{dC}{dt} = ?$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

← 2pts – implicit diff'tiation of distance

$$2(300)(v_A(2)) + 2(400)(v_B(2)) = 2(500)\left(\frac{dC}{dt}\right)$$

$$2(300)(100) + 2(400)(-5(2)^2+60(2)+25) = 2(500)\frac{dC}{dt}$$

$$3(100) + 4(-20+120+25) = 5\frac{dC}{dt}$$

$$300 + 4(125) = 5\frac{dC}{dt}$$

$$800 = 5\frac{dC}{dt}$$

$$dC/dt = 160 \text{ m/min}$$

1pt – answer

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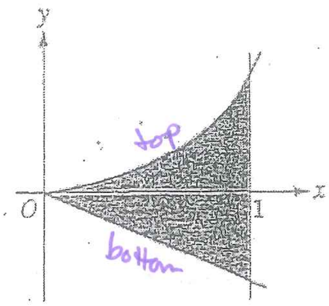
NO CALCULATOR ALLOWED



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area of } R = \int_0^1 (xe^{x^2} - -2x)\,dx$$ 1pt - integrand

$$= \int_0^1 xe^{x^2}\,dx + \int_0^1 2x\,dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,du$
$u(0) = 0$
$u(1) = 1$

$$= \int_0^1 \frac{1}{2}e^u \cdot du + (x^2)\Big|_0^1$$ 1pt - antiderivative

$$= \frac{1}{2}e^u\Big|_0^1 + 1^2 - 0$$

$$= \frac{1}{2}e^1 - \frac{1}{2}e^0 + 1$$ 1pt - answer

ok to stop here

$$= \frac{1}{2}e - \frac{1}{2} + 1$$

$$= \frac{1}{2}e + \frac{1}{2}$$

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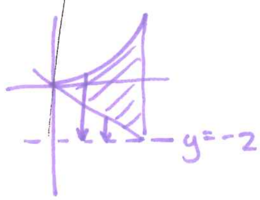
Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.



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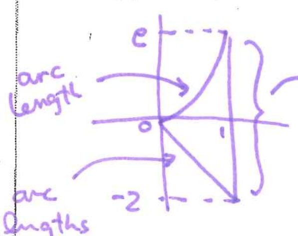
$$V = \pi \int_0^1 \left[(xe^{x^2} - -2)^2 - (-2x - -2)^2 \right] dx$$

2pts - integrand

1pt - limits + π

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.



$e - -2 = e + 2$

1pt - $y' = e^{x^2}(1+2x^2)$

$$\text{Perimeter} = e + 2 + \int_0^1 \sqrt{1 + (e^{x^2} \cdot 1 + xe^{x^2} \cdot 2x)^2}\, dx + \int_0^1 \sqrt{1 + (-2)^2}\, dx$$

1pt - integral

1pt - answer

or

$$P = e + 2 + \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2}\, dx + \int_0^1 \sqrt{5}\, dx$$

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NO CALCULATOR ALLOWED

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

$$\lim_{n\to\infty} \frac{(-1)^{n+2} \cdot \frac{2^{n+1}}{n+1} (x-1)^{n+1}}{(-1)^{n+1} \frac{2^n}{n} (x-1)^n}$$

1pt- set up ratio

$$= \lim_{n\to\infty} \frac{\cancel{(-1)^n} \cdot (-1)^2 \cdot \cancel{2^n} \cdot 2^1 \cdot \cancel{(x-1)^n} (x-1)}{(n+1)} \cdot \frac{n}{(-1)^1 \cancel{(-1)^n}\, \cancel{2^n}\, \cancel{(x-1)^n}}$$

$$= \lim_{n\to\infty} \frac{(-1)^1 \cdot 2(x-1)\, n}{n+1}$$

$$= -2(x-1)$$

1pt- computes limit of ratio

$$|-2(x-1)| < 1$$

$$2|(x-1)| < 1$$

$$|x-1| < \frac{1}{2}$$

$|x-a| < R$

radius of convergence : $\frac{1}{2}$

$$R = \frac{1}{2}$$

1pt- gets R

Continue problem 6 on page 21.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$f(x) = (-1)^2 \cdot \frac{2^1}{1}(x-1)^1 + (-1)^3 \cdot \frac{2^2}{2}(x-1)^2 + (-1)^4 \cdot \frac{2^3}{3}(x-1)^3 + \dots + (-1)^{n+1} \cdot \frac{2^n}{n}(x-1)^n$

$f(x) = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 + \dots + (-1)^{n+1} \cdot \frac{2^n}{n}(x-1)^n$

$f'(x) = 2 - 4(x-1) + 8(x-1)^2 + \dots + (-1)^{n+1} \cdot \frac{2^n}{n} \cdot n(x-1)^{n-1}$

2pt- 3 nonzero terms
1pt- general term

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x - 1| < R$. Use this function to determine f for $|x - 1| < R$.

Look @
$f'(x) \Rightarrow a_1 = 2$ (1st term)
common ratio,
multiplying by -2 and $(x-1)$

$\therefore$, geo series where $a_1 = 2$, $r = -2(x-1)$,

$f'(x) = \frac{a_1}{1-r}$

$f'(x) = \frac{2}{1-(-2(x-1))}$ ← 1pt - $f'(x)$

$= \frac{2}{1+2(x-1)}$

$f'(x) = \frac{2}{2x-1}$

$\int f'(x)\,dx = \int \frac{2}{2x-1}\,dx$

$f(x) = 2\int \frac{1}{2x-1}\,dx$

1pt- antiderive → $= 2\left(\frac{1}{2}\ln|2x-1|\right) + C$

$f(x) = \ln|2x-1| + C$

$f(1) = \ln|2(1)-1| + C$
$0 = \ln 1 + C$
$0 = C$

$f(x) = \ln|2x-1|$ ← 1pt- $f(x)$

for $|x-1| < \frac{1}{2}$

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