NO CALCULATOR ALLOWED

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

$\rightarrow \frac{v(b)-v(a)}{b-a}$

(a) Use the data in the table to estimate the value of $v'(16)$.

$$v'(16) = \frac{v(20)-v(12)}{20-12}$$

$$= \frac{240-200}{20-12}$$

$$= 5 \text{ m/min}^2$$

1pt: approximation (units not required)

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

$$\int_0^{40} |v(t)|\,dt = 150(16) + 220(4) + 240(8) + 200(12)$$

$$= 7600 \text{ meters}$$

← ok to stop here

← simplify not required

1pt: Right sum
1pt: approx.

$\int_0^{40} |v(t)|\,dt$ is the total distance Johanna jogs, in meters, from $t = 0$ to 40 minutes

1pt: explanation (with units)

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Continue problem 3 on page 15.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by velocity → $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

→ $a(t) = v'(t)$

Bob $a(t) \rightarrow B'(t) = 3t^2 - 12t$

Bob's acceleration @ $t=5$ $= B'(5)$

$= 3(5)^2 - 12(5)$ ← ok to stop here

$= 15\,m/min^2$

1pt: uses $B'(t)$

1pt: answer (units optional)

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

$\frac{1}{b-a}\int_a^b v(t)$

Bob avg velocity $= \frac{1}{10-0}\int_0^{10} B(t)\,dt$

$= \frac{1}{10}\int_0^{10} (t^3 - 6t^2 + 300)\,dt$

$= \frac{1}{10}\left(\frac{1}{4}t^4 - 2t^3 + 300t\right)\Big|_0^{10}$

$= \frac{1}{10}\left(\frac{1}{4}(10)^4 - 2(10)^3 + 300(10) - 0\right)$ ← ok to stop here

$= \frac{1}{10}\left(\frac{10000}{4} - 2(1000) + 3000\right)$

$= \frac{1000}{4} - 200 + 300$

$= 250 - 200 + 300$

$= 350\ m/min$

1pt: integral (w/limits)

1pt: antiderivative

1pt: answer (units optional)

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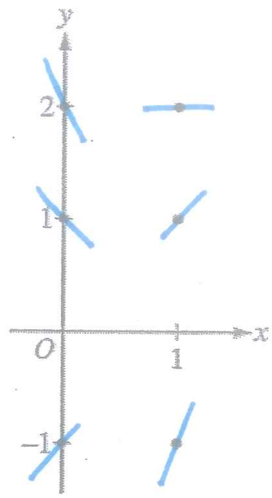
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

y \ x	0	1
2	-2	0
1	-1	1
-1	1	3



1pt: slopes where x=0

1pt: slopes where x=1

2nd derivative

f" >0 or <0 in Quad II

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in $\boxed{\text{Quadrant II}}$. Give a reason for your answer.

$\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$

$= 2 - (2x - y)$

$= 2 - 2x + y$

1pt: $\frac{d^2y}{dx^2}$

In Quad II, $x < 0$ and $y > 0$

$\frac{d^2y}{dx^2} > 0$ in quad II,

$\therefore$, all solution curves are concave up. in Quad II.

1pt: concave up w/ reason

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Continue problem 4 on page 17.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

→ crit # changes pos to neg or neg to pos

crit #, if $\left.\frac{dy}{dx}\right|_{(2,3)} = 0$ or DNE

$\left.\frac{dy}{dx}\right|_{(2,3)} = 2(2) - 3$

$= -1$

1pt: consider $\left.\frac{dy}{dx}\right|_{(2,3)}$

f does not have rel. max nor min @ $x=2$

b/c $\left.\frac{dy}{dx}\right|_{(2,3)} \neq 0$ or DNE... :)

1pt: answer w/ reason

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$

$y = mx + b$

$\frac{dy}{dx} = m$

$2x - y = m$

$2x - (mx + b) = m$

$2x - mx - b = m$

1pt: $\frac{dy}{dx} = m$

1pt: sets $2x - y = m$

$2 - m = 0 \qquad -b = m$

$2 = m$

$-b = 2$

$b = -2$

1pt: answer

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5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}.$$

(a) Let $\boxed{k = 3}$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

$y - y_1 = m(x - x_1)$

$f(4) = \frac{1}{4^2 - 3(4)}$

$= \frac{1}{4}$

$y - \underset{f(4)}{y_1} = \underset{f'(4)}{m}(x - \overset{4}{x_1})$

$\boxed{y - \frac{1}{4} = -\frac{5}{16}(x - 4)}$

1pt: slope

1pt: tangent line

$f'(x) = \frac{3 - 2x}{(x^2 - 3x)^2}$

$f'(4) = \frac{3 - 2(4)}{(4^2 - 3(4))^2}$

$= \frac{-5}{16}$

(b) Let $\boxed{k = 4}$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

$\hookrightarrow$ crit #?

$f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}$

$f'(2) = \frac{4 - 2(2)}{(2^2 - 4(2))^2} = 0$

$f'(2) = 0$, so $x = 2$ is crit #

1pt: considers $f'(2)$

f' + | −

0 2 5

1pt: answer w/ justification

f has rel. max @ $x = 2$ b/c

f' changes from pos to neg @ $x = 2$

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Continue problem 5 on page 19.

(c) Find the value of k for which f has a critical point at $x = -5$.

$\hookrightarrow f' = 0$ or DNE

$$f'(x) = \frac{k-2x}{(x^2-kx)^2} = 0$$

$$f'(-5) = \frac{k-2(-5)}{((-5)^2-k(-5))^2} = \frac{k+10}{(25+5k)^2} = 0$$

$$k = -10$$

OR DNE
$25 + 5k = 0$
$5k = -25$
$k = -5$

1pt: answer

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f.

Find $\int f(x)\,dx$.

$$\frac{1}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6}$$

$1 = A(x-6) + Bx$

$1 = Ax - 6A + Bx$

$0 = A + B \qquad 1 = -6A$

$0 = -\frac{1}{6} + B \qquad -\frac{1}{6} = A$

$\frac{1}{6} = B$

$$\boxed{\frac{1}{x^2-6x} = \frac{-1/6}{x} + \frac{1/6}{x-6}}$$

2pts: partial fraction decomposition

$$\int f(x)\,dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x-6}\right) dx$$

$$= \frac{1}{6}\int\left(-\frac{1}{x} + \frac{1}{x-6}\right) dx$$

$$= \boxed{\frac{1}{6}\left(-\ln|x| + \ln|x-6|\right) + C}$$

OR $\frac{1}{6}\left(\ln\left|\frac{x-6}{x}\right|\right) + C$

2pts: general antiderivative

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NO CALCULATOR ALLOWED

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^n}{n+1} \cdot x^{n+1}}{\frac{(-3)^{n-1}}{n} \cdot x^n} \right|$$

1pt: set up ratio

$$= \lim_{n\to\infty} \left| \frac{(-3)^n}{n+1} x^{n+1} \cdot \frac{n}{(-3)^{n-1} \cdot x^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{x}{(-3)^{-1}} \cdot \frac{n}{n+1} \right)$$

$$= \left| \frac{x}{(-3)^{-1}} \right| = |-3x|$$

1pt: computes limit of ratio

$$|-3x| < 1$$

$$3|x| < 1$$

$$|x| < \frac{1}{3}$$

$$R = \frac{1}{3}$$

1pt: radius of convergence

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Continue problem 6 on page 21.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$$f(x) = x - \frac{3}{2}x^2 + 3x^3 + \cdots + \frac{(-3)^{n-1}x^n}{n} + \cdots$$

$$f'(x) = 1 - 3x + 9x^2 - 27x^3$$

2pts: 1st 4 nonzero terms

$f'(x)$ is geometric

$\frac{a}{1-r}$

$r = -3x$

$$f'(x) = \frac{1}{1-(-3x)}$$

$$= \frac{1}{1+3x}$$

1pt: rational function

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

1pt: 1st 4 nonzero terms for e^x

$$g(x) = e^x f(x)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 + \cdots\right)$$

$$= x - \frac{3}{2}x^2 + 3x^3 + \cdots + x^2 - \frac{3}{2}x^3 + \cdots + \frac{x^3}{2!} + \cdots$$

$$T_3(x) = x - \frac{1}{2}x^2 + 2x^3$$

2pts: Taylor polynomial

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