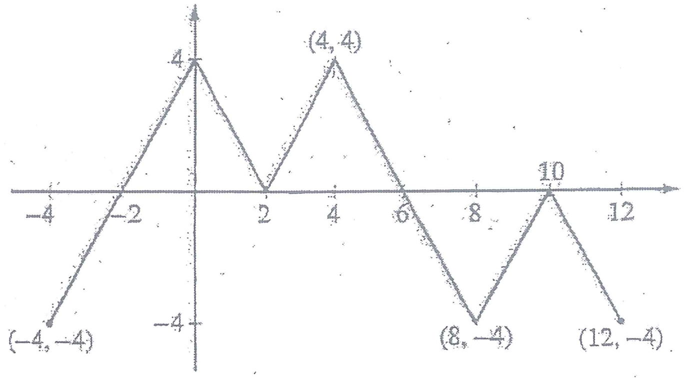


Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

↳ g' pos, neg ↳ g' neg, pos

$g(x) = \int_2^x f(t)\,dt$

$g'(x) = f(x)$

g' − | −
10

1pt: answer w/ reason

g has neither @ $x = 10$ b/c g' does not change signs @ $x = 10$.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

↳ g'' changes signs

$g''(x) = f'(x)$

g'' + | −
4

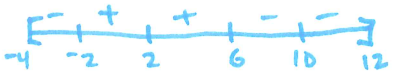
1pt: answer w/ reason

g has inf. pt @ $x = 4$ b/c g'' changes signs @ $x = 4$

Continue problem 3 on page 15.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers. → check rel. min, rel max and endpts.

$g' = f$



↳ g has rel. max @ x = 6 b/c g' changes from pos to neg @ x = 6

1pt: considers x = −2 and x = 6

g has rel. min @ x = −2 b/c g' changes from neg to pos @ x = −2

crit #s

$g(-2) = \int_2^{-2} f(t)\,dt$

$= -\int_{-2}^{2} f(t)\,dt = -\frac{1}{2}(4)(4) = -8$

$g(6) = \int_2^{6} f(t)\,dt = \frac{1}{2}(4)(4) = 8$

endpts

$g(-4) = \int_2^{-4} f(t)\,dt$

$= -\int_{-4}^{2} f(t)\,dt = -\left[-\frac{1}{2}(2)(4) + 8\right]$

$= -[4] = -4$

$g(12) = \int_2^{12} f(t)\,dt = -\frac{1}{2}(2)(4) = -4$

1pt: considers x = −4 and x = 12 (endpts)

2pts: answer w/ justification

abs max value is 8, abs min value is −8.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x) \le 0$ on $[-4, 2] \cup [10, 12]$

2 pts: intervals

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$$

$$= 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$$

$$= 2x - \frac{1}{2}x^2 + \frac{1}{4}y$$

2pts: $\frac{d^2y}{dx^2}$ in terms of $x + y$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$\frac{dy}{dx} = 0$, crit #

$\frac{d^2y}{dx^2} > 0$ min ☺

$\frac{d^2y}{dx^2} < 0$ max

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2}(8)$$

$$= 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2}(-2)^2 + \frac{1}{4}(8)$$

$$= -4 - 2 + 2$$

$$= -4$$

2 pts: conclusion w/ reason

f has rel. max @ $(-2, 8)$ b/c $\left.\frac{dy}{dx}\right|_{(-2,8)} = 0$ and

$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} < 0$

Continue problem 4 on page 17.

Do not write beyond this border.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

$$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right) = \frac{g(-1) - 2}{3(-1+1)^2} = \frac{2-2}{3(0)^2} = \frac{0}{0}$$

L'Hôpital

2pts: L'Hôpital's rule

$$\lim_{x \to -1} \frac{g'(x)}{6(x+1)} = \frac{g'(-1)}{6(-1+1)} = \frac{(-1)^2 - \frac{1}{2}(2)}{6(0)} = \frac{0}{0}$$

L'Hôpital again,

$$\lim_{x \to -1} \frac{g''(x)}{6} = \frac{g''(-1)}{6} = \frac{2(-1) - \frac{1}{2}(-1)^2 + \frac{1}{4}(2)}{6}$$

$$= \frac{-2}{6}$$

$$= -\frac{1}{3}$$

1pt: answer

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

	$\frac{dy}{dx}$	$\frac{dy}{dx}\Delta x$	$\frac{dy}{dx}\Delta x + y$
$(0, 2)$	-1	$-1(\frac{1}{2}) = -\frac{1}{2}$	$-\frac{1}{2} + 2 = \frac{3}{2}$
$(\frac{1}{2}, \frac{3}{2})$	$-\frac{1}{2}$	$-\frac{1}{2}(\frac{1}{2}) = -\frac{1}{4}$	$-\frac{1}{4} + \frac{3}{2} = \frac{5}{4}$

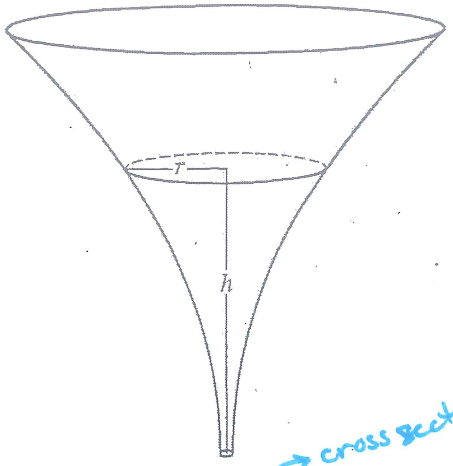
1pt: Euler's method

$$h(1) \approx \frac{5}{4}$$

1pt : answer

Unauthorized copying or reuse of any part of this page is illegal.

NO CALCULATOR ALLOWED



→ cross sections for volume!

5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $\boxed{r = \frac{1}{20}\left(3 + h^2\right)}$, where $\underline{0 \le h \le 10}$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

↳ $\frac{1}{b-a}\int_a^b R$

$$\text{average value} = \frac{1}{10-0}\int_0^{10} r(h)\,dh$$

1pt: integral

$$= \frac{1}{10}\int_0^{10} \frac{1}{20}(3+h^2)\,dh$$

$$= \frac{1}{200}\int_0^{10} (3+h^2)\,dh$$

$$= \frac{1}{200}\left(3h + \frac{1}{3}h^3\right)\Big|_0^{10}$$

1pt: antiderivative

$$= \frac{1}{200}\left(3\cdot 10 + \frac{1}{3}(10)^3 - 0\right)$$

$$= \frac{1}{200}\left(30 + \frac{1000}{3}\right)$$

1pt: answer

Do not write beyond this border.

Continue problem 5 on page 19.

(b) Find the volume of the funnel.

Area of cross section $= \pi r^2$

$= \pi (r(h))^2$

Volume $= \int_0^{10} \pi \left(\frac{1}{20}(3+h^2)\right)^2 dh$ 1pt: integrand

$= \left(\frac{1}{20}\right)^2 \pi \int_0^{10} (3+h^2)^2 dh$

$= \left(\frac{1}{20}\right)^2 \pi \int_0^{10} (9+6h^2+h^4)\, dh$

$= \left(\frac{1}{20}\right)^2 \pi \left(9h+2h^3+\frac{1}{5}h^5\right)\Big|_0^{10}$ 1pt: antidrivative

$= \left(\frac{1}{20}\right)^2 \pi \left(9\cdot 10+2(10)^3+\frac{1}{5}(10)^5-0\right)$ 1pt: answer

---

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$h=3$ $\frac{dr}{dt} = -\frac{1}{5}$ inch/sec $\frac{dh}{dt} = ?$

$r = \frac{1}{20}(3+h^2)$

$\frac{dr}{dt} = \frac{1}{20}\left(0+2h\frac{dh}{dt}\right)$ 2pts: chain rule

$-\frac{1}{5} = \frac{1}{20}\left(2\cdot 3\cdot\frac{dh}{dt}\right)$

$-4 = 6\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{2}{3}$ in/sec 1pt: answer

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$f(x) \approx T(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \dots$$

$$= 1 + \frac{1}{2}(x-1) + \frac{1}{4}\cdot\frac{1}{2!}(x-1)^2 + \frac{-(2!)}{2^3}\cdot\frac{1}{3!}(x-1)^3 + \dots$$

$$\dots + \frac{(-1)^n(n-1)!}{2^n}\cdot\frac{1}{n!}(x-1)^n + \dots$$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{4\cdot 2!} - \frac{1}{2^3\cdot 3}(x-1)^3 + \dots + \frac{(-1)^n}{2^n\cdot n}(x-1)^n + \dots$$

1pt: 1st 2 terms
1pt: 3rd term
1pt: 4th term
1pt: general term

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

1pt: identifies both endpts

check endpts:

@ $x = -1$

$$\sum \frac{(-1)^n(-2)^n}{2^n\cdot n}$$

$$= \sum \frac{2^n}{2^n\cdot n}$$

$$= \sum \frac{1}{n}$$

diverges by p-series test

@ $x = 3$

$$\sum \frac{(-1)^n(2)^n}{2^n\cdot n}$$

$$= \sum \frac{(-1)^n}{n}$$

converges by Alt. Series Test

1pt: analysis interval of convergence

$-1 < x \leq 3$ ← interval of convergence

Do not write beyond this border.

Continue problem 6 on page 21.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$f(1.2) \approx T(1.2)$

$\approx 1 - \frac{1}{2}(1.2-1) + \frac{1}{4 \cdot 2!}(1.2-1)^2$

1pt: approx

Do not write beyond this border.

Do not write beyond this border.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$|R_2(x)| \leq \left| \frac{\max f'''(c)}{3!} \right| |x-1|^3$

$\leq \frac{\frac{2!}{2^3}}{3!}(1.2-1)^3$

$\leq \frac{1}{2^3 \cdot 3}(.2)^3$

$\leq \frac{1}{24}\left(\frac{2}{10}\right)^3$

$\leq \frac{1}{24} \cdot \frac{8}{1000}$

$\leq \frac{1}{3000} \leq \frac{1}{1000}$

1pt: error form
1pt: analysis

Since $f(1.2)$ is alt. series
w/ u_n dec to zero,
error approximation
for $f(1.2)$ is less than
4th term in omitted series
$|\max f'''(c)| = \frac{2!}{2^3}$

GO ON TO THE NEXT PAGE.