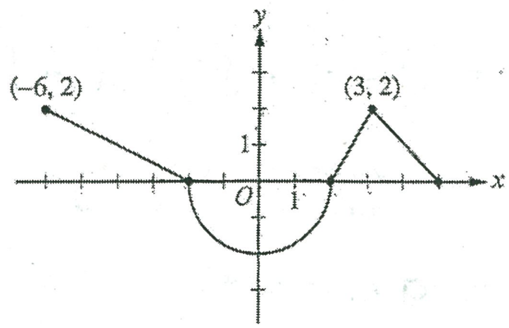
NO CALCULATOR ALLOWED



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

$f(-6) = f(-2) + \int_{-2}^{-6} f'(t)\,dt$

$= 7 - \int_{-6}^{-2} f'(t)\,dt$

$= 7 - \frac{1}{2}(4)(2)$ ← ok to stop here →

$= 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(t)\,dt$

$= 7 + -\frac{1}{2}\pi(2)^2 + \frac{1}{2}(3)(2)$

$= 7 - 2\pi + 3$

$= 10 - 2\pi$

1pt → uses initial condition
1pt → $f(-6)$
1pt → $f(5)$

(b) On what intervals is f increasing? Justify your answer.

f inc on $[-6,-2) \cup (2,5)$ b/c $f' > 0$ on those intervals

2pts → answer w/ reason

Do not write beyond this border.

Continue question 3 on page 15.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

$f'=0$ @ $x=-2, x=2$ → rel min + endpts

f': $+$ | -2 | $-$ | 2 | $+$

↳ f has rel. min @ $x=2$ b/c f' changes from neg to pos @ $x=2$

$f(2) = 7 + \int_{-2}^{2} f'(x)\,dx$

$= 7 + -\frac{1}{2}\pi(2)^2$

$= 7 - 2\pi$

1pt - considers $x=2$

from part (a) → $f(-6) = 3$

$f(5) = 10 - 2\pi$

abs min value is $7 - 2\pi$

1pt - answer w/ justification

Do not write beyond this border.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

$f''(-5) = \frac{2-0}{-6--2} = \frac{-2}{4} = -\frac{1}{2}$

1pt → $f''(-5)$

$f''(3)$ DNE b/c $\lim_{x\to 3^-} f''(x) \neq \lim_{x\to 3^+} f''(x)$

$2 \neq -1$

1pt → $f''(3)$ DNE w/ explanation

NO CALCULATOR ALLOWED

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$y - y_1 = m(x - x_1)$

$y - 91 = -16(x - 0)$

1 pt → tangent line → $H - 91 = -16(t - 0)$

$\left.\frac{dH}{dt}\right|_{t=0} = -\frac{1}{4}(91 - 27)$

$= -\frac{1}{4}(64)$ 1 pt → slope

$= -16$

$H - 91 = -16(3 - 0)$

1 pt → approximation

$H = -16(3) + 91$ ← ok to stop here

$H = -48 + 91$

$H(3) = 43\ °C$ ← units optional ☺

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(H - 27)$

$\frac{d^2H}{dt^2} = -\frac{1}{4}\left(\frac{dH}{dt}\right)$

$= -\frac{1}{4}\left(-\frac{1}{4}(H - 27)\right)$

$= \frac{1}{16}(H - 27)$

1 pt → underestimate w/reason

→ $H > 27$ when $t > 0$, $\therefore \frac{d^2H}{dt^2} > 0$ @ $t = 3$

Part (a) is an underestimate of temp of potato @ $t=3$ b/c $\frac{d^2H}{dt^2} > 0$ @ $t=3$

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $\boxed{G(0) = 91}$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

initial condition

↳ find original function → antiderivative

→ find $G(3)$

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$\int \frac{1}{(G-27)^{2/3}}\, dG = \int -\, dt$$ 1pt → separate variables

$$\int (G-27)^{-2/3}\, dG = -\int dt$$

$u = G-27$

$\frac{du}{dG} = 1$

$du = dG$

$$\int u^{-2/3}\, du = -t + c$$

$$3u^{1/3} = -t + c$$

$$3(G-27)^{1/3} = -t + c$$ 1pt → antiderivatives

1pt → "+c" and initial condition

$$3(91-27)^{1/3} = 0 + c$$

$$3(64)^{1/3} = c$$

$$3(4) = c$$

$$12 = c$$

$$3(G-27)^{1/3} = -t + 12$$ 1pt → equation w/ G & t

$$(G-27)^{1/3} = \frac{-t+12}{3}$$

$$G - 27 = \left(\frac{-t+12}{3}\right)^3$$

$$G = \left(\frac{-t+12}{3}\right)^3 + 27$$

1pt → G(t) and G(3)

$$G(3) = \left(\frac{-3+12}{3}\right)^3 + 27 = 54\ °C$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

$$f(x) = 3(2x^2 - 7x + 5)^{-1}$$

$$f'(x) = (4x - 7) \cdot -3(2x^2 - 7x + 5)^{-2}$$

$$f'(3) = (4 \cdot 3 - 7) \cdot -3(2 \cdot 3^2 - 7 \cdot 3 + 5)^{-2}$$

$$= -15(18 - 21 + 5)^{-2}$$

$$= \frac{-15}{4}$$

2pts → $f'(3)$

$\to f' = 0$ or DNE

(b) Find the x-coordinate of each critical point of f in the interval $\boxed{1 < x < 2.5}$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

$$f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2}$$

$$= \frac{-3(4x - 7)}{[(2x - 5)(x - 1)]^2}$$

not in interval $(1, 2.5)$

not in interval $(1, 2.5)$

crit pts @ $x = 7/4$, ~~$x = 5/2$~~, ~~$x = 1$~~

$f' = 0$ … ☺ … f' DNE

1pt → crit #

f': + | −

$\cancel{1}$ 1.5 $7/4$ 2 $\cancel{5/2}$

1pt → answer w/ reason

f has rel. max @ $x = 7/4$ b/c f' changes from pos to neg @ $x = 7/4$

Continue question 5 on page 19.

(c) Using the identity that $\frac{3}{2x^2-7x+5} = \frac{2}{2x-5} - \frac{1}{x-1}$, evaluate $\int_5^{\infty} f(x)\,dx$ or show that the integral diverges.

$$\int_5^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_5^{a} \left(\frac{2}{2x-5} - \frac{1}{x-1}\right) dx$$

$$= \lim_{a\to\infty} \left(2\cdot \tfrac{1}{2}\ln|2x-5| - \ln|x-1|\right)\Big|_5^a$$ 1pt → antiderivative

$$= \lim_{a\to\infty} \left(\ln\left|\frac{2x-5}{x-1}\right|\right)\Big|_5^a$$ 1pt → limit expression

$$= \lim_{a\to\infty} \left(\ln\left|\frac{2a-5}{a-1}\right| - \ln\left(\frac{5}{4}\right)\right)$$

$$= \ln\left(\lim_{a\to\infty}\left|\frac{2a-5}{a-1}\right|\right) - \ln\left(\frac{5}{4}\right)$$

$$= \ln 2 - \ln\left(\frac{5}{4}\right)$$ 1pt → answer

$$= \ln\left(8/5\right)$$

(d) Determine whether the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 \cancel{-7n+5}}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

$\sum_{n=5}^{\infty} \frac{3}{2n^2-7n+5}$ converges b/c $\int_5^{\infty} f(x)\,dx$ converges by integral test

2pts → answer w/ condition

||OR||

$a_k = \frac{2}{2n^2-7n-5}$ and $b_k = \frac{1}{x^2}$

$a_k > b_k$, and since $\sum b_k$ converges by p-series, then $\sum a_k$ converges by comparison test

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{IV}(0)}{4!}(x-0)^4 + \dots + \boxed{\frac{f^{(n)}(0)}{n!}(x-0)^n}$$

$$= 0 + 1(x) + \frac{-1f'(0)}{2!}x^2 + \frac{-2f''(0)}{3!}x^3 + \frac{-3f'''(0)}{4!}x^4 + \dots + \frac{-nf^{(n)}(0)}{n!}x^n + \dots$$

$$= x - \frac{-1(1)}{2!}x^2 + \frac{-2((-1)(1))}{3!}x^3 + \frac{-3(-2(-1)(1))}{4!}x^4 + \dots$$

$$= x + \frac{x^2}{2} + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \dots + \frac{(-1)^{2n+1}}{n}x^n + \dots$$

$$= x + \frac{x^2}{2} + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + \frac{(-1)^{2n+1}}{n}x^n + \dots$$

1pt → $f''(0)$ and $f'''(0)$ and $f^{(4)}(0)$

1pt → verify terms

1pt → general term

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}x^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}(1)^n \Rightarrow \sum_{n=1}^{\infty} \left|\frac{(-1)^{2n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges by p-series}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$$

Alt. Series Test

① $\frac{1}{n} > 0$

② $\frac{1}{n+1} > \frac{1}{n}$

③ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$ converges

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}x^n$ converges conditionally @ $x=1$

2pts → converges conditionally w/ reason

Do not write beyond this border.

Do not write beyond this border.

Continue question 6 on page 21.

NO CALCULATOR ALLOWED

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$$g(x) = \int_0^x f(t)\,dt$$

$$= \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \dots + \frac{(-1)^{2n+1}}{n} \cdot \frac{1}{n+1}x^{n+1} + \dots$$

1pt → 2 terms

1pt → ~~remaind~~ remaining terms

1pt ⇒ general term

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

$$\left|P_4\left(\tfrac{1}{2}\right) - g\left(\tfrac{1}{2}\right)\right| \le \frac{|\max f^{(5)}(c)|}{5!}|x-0|^5$$

$$\le \left|\tfrac{-1}{20}\right|\left(\tfrac{1}{2}\right)^5$$

$$\le \frac{1}{160} < \frac{1}{500}$$

1pt → error bound

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.