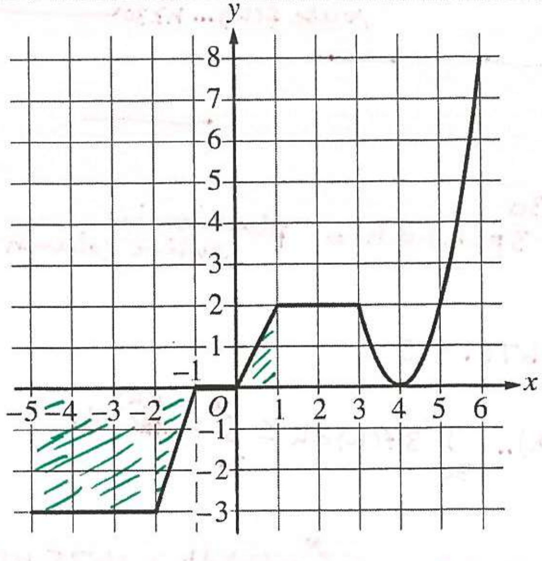


Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $\boxed{g(x) = 2(x-4)^2}$ for $3 \le x \le 6$. → $g = f'$

(a) If $f(1) = 3$, what is the value of $f(-5)$?

$f(-5) = f(1) + \int_1^{-5} f'(x)\,dx$

$= 3 - \int_{-5}^{1} g(x)\,dx$

$= 3 - \left(-3\cdot 3 + -\frac{1}{2}(1)(3) + \frac{1}{2}(1)(2)\right)$ ← ok to stop here

$= 3 + 9 + \frac{3}{2} - 1$

$= 11 + \frac{3}{2} = 25/2$

1pt: integral

1pt: answer

(b) Evaluate $\int_1^6 g(x)\,dx$.

$\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 2(x-4)^2\,dx$

$u = x-4$
$du = dx$

$= 2\cdot 2 + \frac{2}{3}(x-4)^3 \Big|_3^6$

$= 4 + \frac{2}{3}\left((6-4)^3 - (3-4)^3\right)$ ← ok to stop here

$= 4 + \frac{2}{3}(8 --1)$

$= 10$

1pt: split integral @ $x=3$

1pt: antiderivative

1pt: answer

Continue question 3 on page 15.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

↳ $f' > 0$ ↳ $f'' > 0$ or f' inc

$g > 0$ and g' inc

f inc and concave up on $(0,1) \cup (4,6)$

b/c $f' > 0$ and f' inc on those intervals

1pt : intervals

1pt : reason

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

↳ f'' changes signs

$g = f'$

$g' = f''$

$f'' = 0$ on $(-1, 0)$ and $(1, 3)$

and @ $x = 4$

f'' − | + (4)

f has inf pt @ $x = 4$ b/c

f'' changes signs @ $x = 4$

(or f' changs from dec to inc @ $x = 4$)

1pt : answer

1pt : reason

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

↳ $H'>0 \rightarrow H$ inc
$H'<0 \rightarrow H$ dec

$$H'(6) = \frac{H(7)-H(5)}{7-5}$$

$$= \frac{11-6}{7-5}$$ ← ok to stop here

$$= \frac{5}{2}\ \frac{\text{meters}}{\text{year}}$$

1pt: estimate

The height of the tree is increasing $\frac{5}{2}$ m/year @ $t=6$ years

1pt: interpretation w/units

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

IVT or MVT

slope... MVT?

H is cont b/c H is twice-diff'able
H is diff'able b/c H is twice-diff'able

$$H'(t) = \frac{H(5)-H(3)}{5-3}$$

1pt: $\frac{H(5)-H(3)}{5-3}$

$$= \frac{6-2}{2}$$

$$= \frac{4}{2}$$

$$= 2$$

1pt: conclusion w/MVT

∴, by MVT, there must be at least one time t,
on $2<t<10$ s.t. $H'(t)=2$.

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

$\frac{1}{2}(b_1+b_2)(h)$

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

$\frac{1}{b-a}\int_a^b$

1pt: trap sum

1pt: approximation

$$\text{avg height} = \frac{1}{10-2}\int_2^{10} H(t)\,dt$$

$$= \frac{1}{8}\left[\frac{1}{2}(1.5+2)(1)+\frac{1}{2}(2+6)(2)+\frac{1}{2}(6+11)(2)+\frac{1}{2}(11+15)(3)\right]$$

← ok to stop here

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$G=50$

$\frac{dG}{dt}$

derivative

@ $G=50$

$G=50$

$\frac{dx}{dt}=0.03$

2pt: $\frac{d}{dt}(G(x))$

$$G(x)=\frac{100x}{1+x}$$

$$\frac{dG}{dt}=\frac{(1+x)(100\frac{dx}{dt})-100x(\frac{dx}{dt})}{(1+x)^2}$$

need x...

$50=\frac{100x}{1+4}$

$50+50x=100x$

$50=50x$

$1=x$

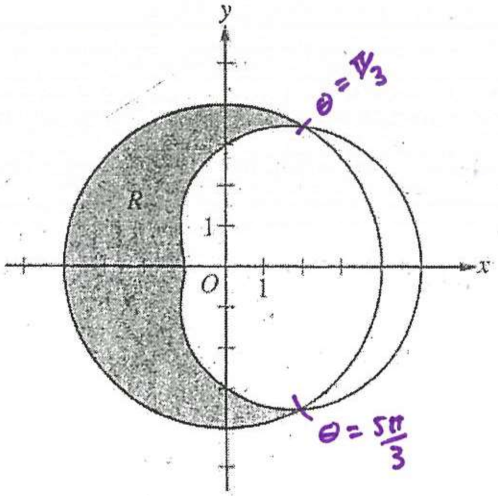
$$\left.\frac{dG}{dt}\right|_{G=50}=\frac{(1+1)(100\cdot 0.03)-100(1)(.03)}{(1+1)^2}$$

← ok to stop here

$$=\frac{100(.03)(2-1)}{4}$$

$$=\frac{3}{4}$$

1pt: answer



5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

$$\text{Area of } R = \frac{1}{2}\int_{\pi/3}^{5\pi/3} (4)^2\, d\theta - \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3+2\cos\theta)^2\, d\theta$$

1pt: constant + limits

2pts: integrand

Continue question 5 on page 19.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

$\hookrightarrow \frac{dy/d\theta}{dx/d\theta} = dy/dx$

$x = r\cos\theta \quad y = r\sin\theta$

$x = (3+2\cos\theta)\cos\theta \quad y = (3+2\cos\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-2\sin\theta) + (3+2\cos\theta)\cos\theta}{\cos\theta(-2\sin\theta) + (3+2\cos\theta)(-\sin\theta)}$$

1 pt: $\frac{dy}{d\theta}$ or $\frac{dx}{d\theta}$

1 pt: $\frac{dy}{dx}$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{\sin\frac{\pi}{2}(-2\sin\frac{\pi}{2}) + (3+2\cos\frac{\pi}{2})(\cos\frac{\pi}{2})}{\cos\frac{\pi}{2}(-2\sin\frac{\pi}{2}) + (3+2\cos\frac{\pi}{2})(-\sin\frac{\pi}{2})}$$

← ok to stop here

1 pt: answer

$$= \frac{-2 + (3+0)(0)}{0(-2) + (3+0)(-1)}$$

$$= \frac{-2}{-3}$$

$$= 2/3$$

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

$\hookrightarrow$ radius ... r

$\frac{d\theta}{dt} = ?$

$\frac{dr}{dt} = 3$ units/sec

$$r = 3 + 2\cos\theta$$

$$\frac{dr}{dt} = -2\sin\theta\frac{d\theta}{dt}$$

1 pt: $\frac{dr}{dt}$ in terms of $\frac{d\theta}{dt}$

$$3 = -2\sin\frac{\pi}{3}\frac{d\theta}{dt}$$

1 pt: $\frac{dr}{dt}$ in terms of θ

$$\frac{d\theta}{dt} = \frac{3}{-2\sin\frac{\pi}{3}}$$

$\frac{d\theta}{dt} \to \frac{\text{radians}}{\text{sec}}$

$$= -\frac{3}{\sqrt{3}}\ \frac{\text{radians}}{\text{sec}}$$

1 pt: answer w/ units

6. The Maclaurin series for $\boxed{\ln(1+x)}$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

$$\ln\left(1+\frac{x}{3}\right) = \frac{x}{3} - \frac{(x/3)^2}{2} + \frac{(x/3)^3}{3} - \frac{(x/3)^4}{4} + \ldots + (-1)^{n+1}\frac{(x/3)^n}{n} + \ldots$$

$$f(x) = x\cdot\ln\left(1+\frac{x}{3}\right) = \frac{x^2}{3} - \frac{x^3}{9\cdot 2} + \frac{x^4}{3^3\cdot 3} - \frac{x^5}{3^4\cdot 4} + \ldots + (-1)^{n+1}\frac{x^{n+1}}{3^n\cdot n} + \ldots$$

1pt: 1st 4 terms

1pt: general term.

Continue question 6 on page 21.

(b) Determine the interval of convergence of the Maclaurin series for *f*. Show the work that leads to your answer.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}}{3^{n+1}x^{n+1}}\cdot\frac{3^n\cdot n}{(-1)^{n+1}x^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-1)x\cdot n}{3(n+1)}\right|$$

$$=\left|\frac{x}{3}\right|$$

$$\left|\frac{x}{3}\right|<1$$

$$-3<x<3$$

1pt: set up ratio

1pt: compute limit of ratio

1pt: radius of convergence

check endpts:

$x=-3$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-3)^{n+1}}{3^n\cdot n}$

$=\sum_{n=1}^{\infty}\frac{3^{n+1}}{3^n\cdot n}$

$=\sum_{n=1}^{\infty}\frac{3}{n}$

diverges, p-series

$x=3$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 3^{n+1}}{3^n\cdot n}$

$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot 3}{n}$

alt. series test ① $\frac{3}{n}>0$

② $\frac{3}{n}>\frac{3}{n+1}$

③ $\lim_{n\to\infty}\frac{3}{n}=0$

converges by a.s.t.

1pt: considers endpts

$\therefore$, interval of convergence: $(-3,3]$

1pt: analysis and interval of convergence

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for *f* about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

$$|P_4(2)-f(2)|\le\frac{|\max f^5(c)|}{5!}(2)^5$$

$$\le\frac{\frac{5!}{3^4\cdot 4}}{5!}\cdot 2^5$$

$$\le\frac{1}{3^4\cdot 4}\cdot 2^5$$

$$-\frac{1}{3^4\cdot 4}=\frac{f^5(0)}{5!}$$

$$\frac{-5!}{3^4\cdot 4}=f^5(0)$$

1pt: use 5th degree term as error bound

1pt: answers