rate enter

1. Fish $\boxed{\text{enter a lake at a rate}}$ modeled by the function E given by $\boxed{E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)}$. Fish $\boxed{\text{leave the lake at a rate}}$ modeled by the function L given by $\boxed{L(t) = 4 + 2^{0.1t^2}}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

$\int$ rate enter

$$\text{\# fish enter lake} = \int_0^5 E(t)\,dt$$

1pt- integral

$$= 153.478$$

$$\approx 153$$

1pt- answer

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

$\frac{1}{b-a}\int_a^b$ rate leave

$$\text{avg \# leave lake} = \frac{1}{5-0}\int_0^5 L(t)\,dt$$

1pt- integral

$$= 6.059$$

1pt- answer

Continue question 1 on page 5.

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

↳ abs max → crit #s & endpts into original equation

fish in lake = ∫(rate enter − rate leave)

$F(t) = \int (E(t) - L(t))\, dt$

rate of fish in lake → $F'(t) = E(t) - L(t)$ rate enter − rate leave

$E(t) - L(t) = 0$

$t = 6.204$

$F(6.204) = \int_0^{6.204} (E(t) - L(t))\, dt = 135.015$

$F(0) = 0$

$F(8) = \int_0^{8} (E(t) - L(t))\, dt = 80.920$

@ $t = 6.204$, there is greatest # of fish in lake

1pt: $E(t) - L(t) = 0$

1pt: answer

1pt: justification

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

↳ rate inc → (rate)' > 0

rate dec → (rate)' < 0

rate fish in lake $= E(t) - L(t)$ ← need derivative of this rate

$E'(t) - L'(t)$

$E'(5) - L'(5) = -10.723$

Rate of change in # of fish is dec @ $t = 5$

b/c $E'(5) - L'(5) < 0$

1pt: considers $E'(5)$ and $L'(5)$

1pt: answer w/ reason

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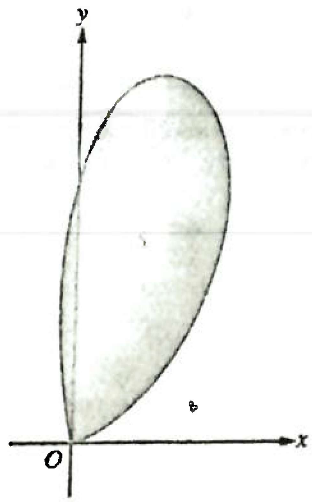


2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

α β

(a) Find the area of S.

$\hookrightarrow \frac{1}{2}\int_\alpha^\beta r^2$

Area of S $= \frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta$

$= 3.534$

1pt: integral
1pt: answer

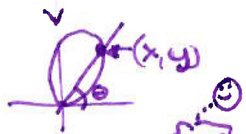
(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

Avg distance $= \frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} r(\theta)\, d\theta$

$= 1.580$

1pt: integral
1pt: answer

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.



$\tan\theta = \frac{x}{y}$ rise/run

$\tan\theta = m$

$\theta = \tan^{-1} m$

1pt: equates polar areas

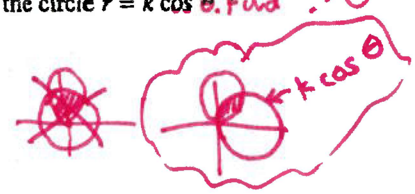
1pt: intersecting w/ m

1pt: equation

$$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2 \, d\theta\right)$$

or $$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2}\int_{\tan^{-1} m}^{\sqrt{\pi}} (r(\theta))^2 \, d\theta$$

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim\limits_{k\to\infty} A(k)$.



$$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\,\sin(\theta^2)\right)^2 d\theta$$

1pt: limits

1pt: answer w/ integral

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