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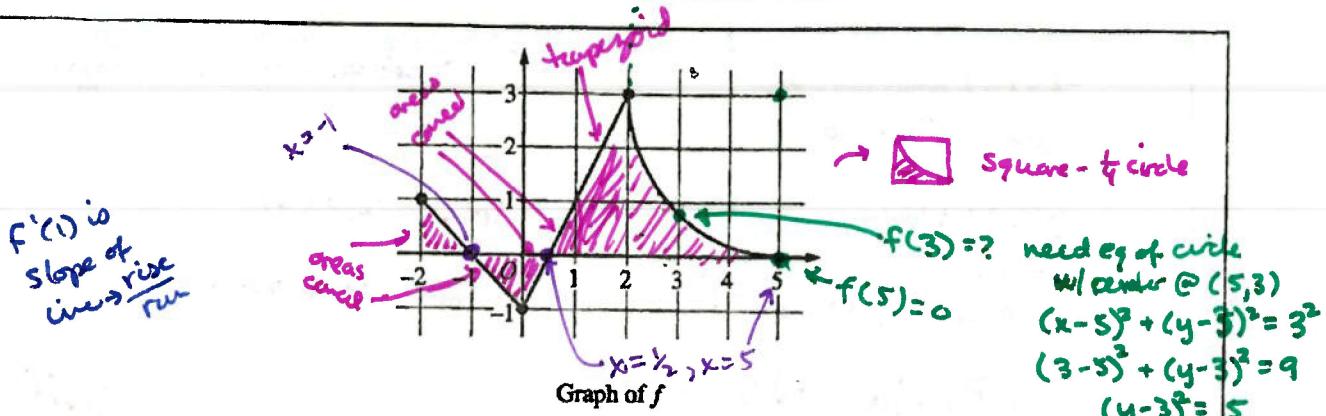
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3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f , consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f .

(a) If $\int_{-6}^5 f(x) dx = 7$, find the value of $\int_{-6}^{-2} f(x) dx$. Show the work that leads to your answer.

$\overbrace{-6 \dots -2}$
 $\overbrace{? \dots 5}$
 graph

$$\begin{aligned}
 \int_{-6}^{-2} f(x) dx &= \int_{-6}^5 f(x) dx - \int_{-2}^5 f(x) dx && \leftarrow 1\text{pt:} \\
 &= 7 - \left[\frac{1}{2}(1+3)(1) + 9 - \frac{1}{4}\pi(3)^2 \right] && \leftarrow \text{ok to stop here} \\
 &= 7 - [2 + 9 - \frac{9\pi}{4}] && \leftarrow 1\text{pt: } \int_{-2}^5 f(x) dx \\
 &= 7 - 11 + \frac{9\pi}{4} && \leftarrow 1\text{pt: answer} \\
 &= -4 + \frac{9\pi}{4}
 \end{aligned}$$

(b) Evaluate $\int_3^5 (2f'(x) + 4) dx$.

$$\begin{aligned}
 \int_3^5 (2f'(x) + 4) dx &= 2 \int_3^5 f'(x) dx + \int_3^5 4 dx \\
 &= 2f(x)|_3^5 + 4x|_3^5 && \leftarrow 1\text{pt: FTC} \\
 &= 2[f(5) - f(3)] + 4(5) - 4(3) \\
 &= 2[0 - (3 - \sqrt{5})] + 4(5) - 4(3) && \leftarrow \text{ok to stop here} \\
 &= -6 + 2\sqrt{5} + 20 - 12 && \leftarrow 1\text{pt: answer} \\
 &= 2 + 2\sqrt{5}
 \end{aligned}$$

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NO CALCULATOR ALLOWED

- (c) The function g is given by $g(x) = \int_{-2}^x f(t) dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

crit #s + endpts in original equation

$$g(x) = \int_{-2}^x f(t) dt$$

$$g'(x) = f(x)$$

$$f(x) = 0$$

$$\text{at } x = -1, x = \frac{1}{2}, x = 5$$

$$1\text{pt: } g'(x) = f(x)$$

$$1\text{pt: crit # } x = -1$$

$$g(-1) = \int_{-2}^{-1} f(t) dt = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$g\left(\frac{1}{2}\right) = \int_{-2}^{\frac{1}{2}} f(t) dt = -\frac{1}{2}\left(\frac{1}{2}\right)(1) = -\frac{1}{4}$$

$$g(5) = \frac{1}{2}(1+3)(1) + 9 - \frac{1}{4}\pi(3)^2 = 11 - \frac{9\pi}{4} \quad \leftarrow \text{abs max value of } g \text{ is } 11 - \frac{9\pi}{4}$$

did this
in part a.

$$g(-2) = 0$$

1 pt: answer w/ justification

- (d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

slope rise over run

$$\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$$

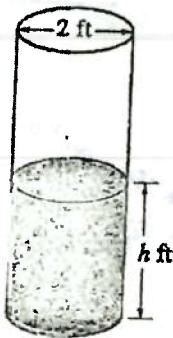
$$= \frac{10 - 3(2)}{1 - \arctan 1} \quad \leftarrow \text{ok to stop here}$$

1 pt: answer

$$= \frac{4}{1 - \frac{\pi}{4}}$$

$$= \frac{16}{4 - \pi}$$

NO CALCULATOR ALLOWED



$$d=2, r=1$$

4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$$h=4$$

$$r=1 \quad r \text{ is constant}$$

for a cylinder

$$V = \pi r^2 h$$

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\frac{dv}{dt} = ?$$

$$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$$

$$\frac{dh}{dt} \Big|_{h=4} = -\frac{1}{10}\sqrt{4}$$

$$1pt: \frac{dv}{dt}$$

1pt: answer w/
units

$$\begin{aligned} & \frac{dv}{dt} = \pi(1)^2\left(-\frac{1}{10}\sqrt{4}\right) \leftarrow \text{ok to stop here}\right. \\ & \left. \frac{dv}{dt} = -\frac{\pi}{5} \frac{ft^3}{sec} \right. \end{aligned}$$

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NO CALCULATOR ALLOWED

- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

rate inc $\rightarrow (\text{rate})' > 0$
 rate dec $\rightarrow (\text{rate})' < 0$

} derivative of rate is 2nd derivative

$$\frac{d^2h}{dt^2} = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}} \frac{dh}{dt}$$

2 pts. $\frac{d^2h}{dt^2}$

$$\left. \frac{d^2h}{dt^2} \right|_{h=3} = -\frac{1}{10} \cdot \frac{1}{2} (3)^{-\frac{1}{2}} \left(-\frac{1}{10} \sqrt{4} \right)$$

> 0

Rate of change of height of water is inc when $h=3$

b/c $\left. \frac{d^2h}{dt^2} \right|_{h=3} > 0$.

1 pt: answer w/
reason

- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .

initial condition

$\hookrightarrow x w/dx, y w/dy$

$$\frac{dh}{dt} = -\frac{1}{10} \sqrt{h}$$

1 pt: separate variables

$$\frac{1}{\sqrt{h}} dh = -\frac{1}{10} dt$$

1 pt: antiderivatives

$$\int h^{-\frac{1}{2}} dh = \int -\frac{1}{10} dt$$

1 pt: "+C" and initial condition

$$2h^{\frac{1}{2}} = -\frac{1}{10} t + C \rightarrow 2h^{\frac{1}{2}} = -\frac{1}{10} t + 2\sqrt{5}$$

1 pt: solves for h

$$2(5)^{\frac{1}{2}} = -\frac{1}{10}(0) + C$$

$$h^{\frac{1}{2}} = \frac{-\frac{1}{10}t + 2\sqrt{5}}{2}$$

$$2\sqrt{5} = C$$

$$h = \left(\frac{-\frac{1}{10}t + 2\sqrt{5}}{2} \right)^2$$

or

$$h = \left(-\frac{1}{20}t + \sqrt{5} \right)^2$$

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NO CALCULATOR ALLOWED

5. Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

- (a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f(x) = (x^2 - 2x + k)^{-1} \quad \rightarrow f' = 6$$

$$\begin{aligned} f'(x) &= (2x-2) \cdot -1(x^2 - 2x + k)^{-2} \\ &= \frac{-(2x-2)}{(x^2 - 2x + k)^2} \end{aligned}$$

$$f'(0) = \frac{-(2(0)-2)}{(0^2 - 2(0) + k)^2}$$

$$6 = \frac{2}{k^2}$$

$$k^2 = \frac{2}{6}$$

$$k = \pm \sqrt{\frac{1}{3}} \rightarrow k = \sqrt{\frac{1}{3}} \text{ or } \frac{1}{\sqrt{3}}$$

1pt: denominator of f'

1pt: f'

1pt: answer

- (b) For $k = -8$, find the value of $\int_0^1 f(x) dx$.

$$f(x) = \frac{1}{x^2 - 2x - 8}$$

$$\int_0^1 \frac{1}{x^2 - 2x - 8} dx$$

$$= \int_0^1 \left(\frac{1}{x-4} - \frac{1}{x+2} \right) dx$$

$$= \frac{1}{6} \int_0^1 \left(\frac{1}{x-4} - \frac{1}{x+2} \right) dx$$

$$= \frac{1}{6} \left[(\ln|x-4| - \ln|x+2|) \right]_0^1$$

$$= \frac{1}{6} [\ln 3 - \ln 3 - (\ln 4 - \ln 2)] \quad \leftarrow \text{stop here}$$

$$= \frac{1}{6} (-\ln 4 + \ln 2) = \frac{1}{6} (-\ln 2^2 + \ln 2) = \frac{1}{6} (-2\ln 2 + \ln 2) = -\frac{1}{6} \ln 2$$

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-4)$$

$$1 = Ax + 2A + Bx - 4B$$

$$A+B=0 \quad 2A-4B=1$$

$$A=-B \quad -2B-4B=1$$

$$A=\frac{1}{6} \quad B=-\frac{1}{6}$$

1pt: partial fraction decomposition

1pt: antiderivative

1pt: answer

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NO CALCULATOR ALLOWED

(c) For $k = 1$, find the value of $\int_0^2 f(x) dx$ or show that it diverges.

$$\begin{aligned}
 \int_0^2 \frac{1}{x^2 - 2x + 1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx \\
 &= \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\
 &= \lim_{a \rightarrow 1^-} \int_0^a \frac{1}{(x-1)^2} dx + \lim_{a \rightarrow 1^+} \int_a^2 \frac{1}{(x-1)^2} dx \\
 &= \lim_{a \rightarrow 1^-} \left(-\frac{1}{x-1} \right) \Big|_0^a + \lim_{a \rightarrow 1^+} \left(-\frac{1}{x-1} \right) \Big|_a^2 \\
 &= \lim_{a \rightarrow 1^-} \left(-\frac{1}{a-1} - 1 \right) + \lim_{a \rightarrow 1^+} \left(-1 + \frac{1}{a-1} \right)
 \end{aligned}$$

DNE, \therefore integral divergesimproper
@ $x=1$ 1 pt: improper
integral

1 pt: antiderivative

1 pt: answer w/
redundant

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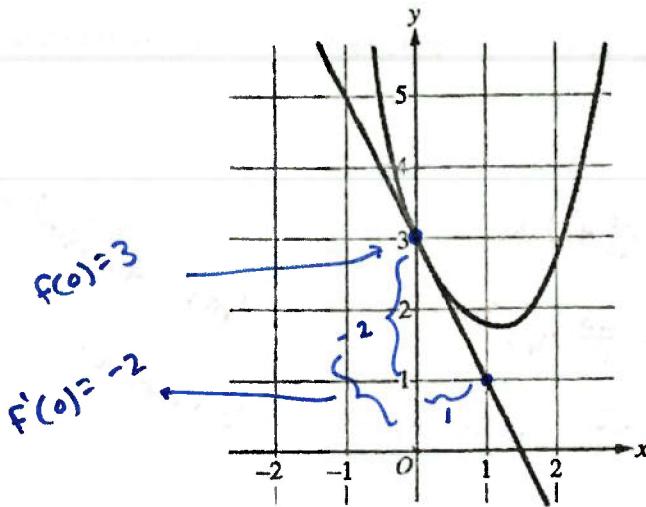
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n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x . A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

- (a) Write the third-degree Taylor polynomial for f about $x = 0$.

$$f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$T_3(x) = f(x) \approx 3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3$$

$$\text{or } 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$$

(pt: 2 terms
pt: remaining terms)

- (b) Write the first three nonzero terms of the Maclaurin series for e^x . Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

$$e^x f(x) \approx (1 + x + \frac{x^2}{2!})(3 - 2x + \frac{3}{2!}x^2 - \frac{\frac{23}{2}}{3!}x^3)$$

$$\approx 3 + 3x + \frac{3x^2}{2!} - 2x - 2x^2 - \frac{2x^3}{2!} + \frac{3}{2!}x^2 + \dots$$

$$\approx 3 + x + \frac{3}{2}x^2 - 2x^2 + \frac{3}{2}x^2 + \dots$$

(pt: 3 terms for e^x
pt: 3 terms for $f(x)$)

(pt: 3 terms for $e^x f(x)$)

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$$T_2(x) = 3 + x + x^2$$

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Continue question 6 on page 2

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- (c) Let h be the function defined by $h(x) = \int_0^x f(t) dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$$\begin{aligned}
 h(x) &= \int_0^x f(t) dt \\
 &= \int_0^x \left(3 - 2t + \frac{3}{2}t^2 + \frac{23}{12}t^3\right) dt \\
 &= 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{12} \cdot \frac{1}{4}x^4 \\
 h(1) &= 3 - 1 + \frac{1}{2} - \frac{23}{12} \cdot \frac{1}{4} \quad \text{← ok to stop here} \\
 &= \frac{97}{48}
 \end{aligned}$$

1 pt: antiderivative
 1 pt: answer

- (d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x . It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

altseries
 error bound
 qualifications
 met ... need
 4th degree term
 for $f(x)$

$$\frac{f^{(4)}(x)}{4!} x^4 = \frac{54}{4!} x^4$$

(pt: use 4th degree
 term for
 term for $h(x)$)
 (pt: uses 4th
 term omitted)

~~$\approx h(1) - \text{approx}$~~

$$\text{need } h(1) - \text{approx} \rightarrow \int_0^1 \frac{54}{4!} x^4 dx = \frac{9}{20} x^5 \Big|_0^1 = \frac{9}{20}$$

(pt: error
 bound)

$$|h(1) - \text{approx}| \leq \frac{9}{20} = 0.45$$