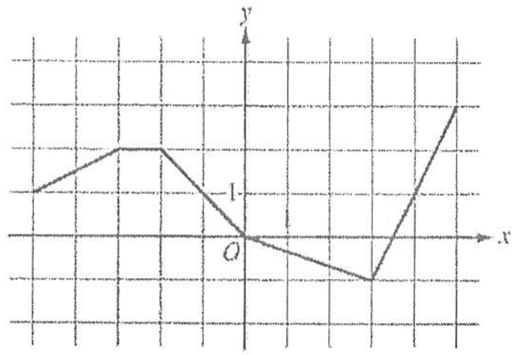
Finding the Errors in Student Work

Given the following AP Free-Response Question, view the students' work and determine where the students made any errors, if any.

2017 AP® CALCULUS AB FREE-RESPONSE QUESTIONS

x	$g(x)$	$g'(x)$
-5	10	-3
-4	5	-1
-3	2	4
-2	3	1
-1	1	-2
0	0	-3



Graph of h

6. Let f be the function defined by $f(x) = \cos(2x) + e^{\sin x}$.

   Let g be a differentiable function. The table above gives values of g and its derivative g' at selected values of x.

   Let h be the function whose graph, consisting of five line segments, is shown in the figure above.

   (a) Find the slope of the line tangent to the graph of f at $x = \pi$.

   (b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

   (c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

Work for Student 1

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$f(x) = \cos(2x) + e^{\sin x}$

$f'(x) = -2\sin(2x) + (\cos x)\, e^{\sin x}$

$f'(\pi) = -2\sin(2\pi) + (\cos\pi)\, e^{\sin\pi}$

$f'(\pi) = -2(0) + (-1)e^{(0)}$

$f'(\pi) = 0 + -1(1) = -1$

correct work, no errors

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$k(x) = h(f(x))$

$k'(x) = h'(f(x))\, f'(x)$

$k'(\pi) = h'(f(\pi))\, f'(x)$

should be π

$k'(\pi) = [h'(2)](-1)$

$k'(\pi) = (-\frac{1}{3})(-1)$

$f(\pi) = \cos(2\pi) + e^{\sin\pi}$

$f(\pi) = 1 + e^{0} = 1 + 1 = 2$

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$m(x) = g(-2x) \cdot h(x)$

$m'(x) = -2g'(-2x) \cdot h(x) + h'(x) \cdot g(-2x)$

$m'(2) = -2g'(-4) \cdot h(2) + h'(2) \cdot g(-4)$

$m'(2) = -2[-1] \cdot \frac{2}{3} + (\frac{1}{3}) \cdot 5$

negative missing

$m'(2) = \frac{8}{3} + \frac{5}{3} = \frac{13}{3}$

should be $-\frac{4}{3}$

correct answer is: -3

Work for Student 2

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$$f'(x) = -2\sin(2x) + e^{\sin x}\cos(x)$$

$$f'(\pi) = -2\sin(2\pi) + e^{\sin \pi}\cos(\pi)$$

$$= 0 + 1(-1)$$

$$f'(\pi) = \boxed{-1}$$

no errors

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$$k'(x) = h'(f(x))\, f'(x)$$

$$k'(\pi) = h'(f(\pi))\, f'(\pi)$$

never evaluated, need to get values for $f(\pi)$, $f'(\pi)$ ...

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$$m'(x) = g'(-2x)(-2) \cdot h'(x)$$

did not do product rule.

$$m'(2) = g'(-4) - 2 \cdot h'(-2)$$

$$m'(2) = (4 - 2) \cdot -1$$

$$m'(2) = 2 \cdot -1$$

$$\boxed{m'(2) = -2}$$

Work for Student 3

(a) Find the slope of the line tangent to the graph of f at $x = \pi$.

$$f'(x) = -2\sin(2x) + e^{\sin x}\ \frac{1}{\cos(x)}$$

wrong derivative of inside function

$$f'(\pi) = -2\sin(2\pi) + e^{\sin \pi}\ \frac{1}{\cos(\pi)}$$

$$-2\sin(0) + e^{\sin 0}$$

$$= -1$$

got to correct answer by luck.

(b) Let k be the function defined by $k(x) = h(f(x))$. Find $k'(\pi)$.

$$k'(x) = h'(f(x))f'(x)$$

$$k'(\pi) = h'(f(\pi)) \cdot f'(\pi)$$

$$= h'(\cos 2\pi + e^{0}) \cdot -1$$

$$= -h'(1) = +\tfrac{1}{3}$$

Should be 2

correct answer by luck again

(c) Let m be the function defined by $m(x) = g(-2x) \cdot h(x)$. Find $m'(2)$.

$$m'(x) = g'(-2x) \cdot -2 \cdot h(x) - g(-2x) \cdot h'(x)$$

$$m'(2) = g'(-4) \cdot -2 \cdot h(2) - g(-4) \cdot h'(2)$$

$$= -1 \cdot -2 \cdot \tfrac{2}{3} - 5 \cdot -\tfrac{1}{3} = -\tfrac{11}{3} + \tfrac{5}{3}$$

$$= -\tfrac{6}{3} = -2$$

product rule... Should have addition

Should have $-\frac{4}{3}$, not $\frac{11}{3}$

AP Multiple-Choice Exponential & Logarithmic Functions

4. If $f(x) = 7x - 3 + \ln x$, then $f'(1) =$

(A) 4 (B) 5 (C) 6 (D) 7 $\boxed{\text{(E)}}$ 8

$f'(x) = 7 + \frac{1}{x}$

$f'(1) = 7 + \frac{1}{1} = 8$

18. $\lim_{h \to 0} \frac{\ln(4+h) - \ln(4)}{h}$ is ← definition of the derivative where the function is $\ln x$ @ $x = 4$

(A) 0 $\boxed{\text{(B)}}$ $\frac{1}{4}$ (C) 1 (D) e (E) nonexistent

$f'(4) = \lim_{h \to 0} \frac{\ln(4+h) - \ln 4}{h}$, where $f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f'(4) = \frac{1}{4}$

13. If $f(x) = x^2 + 2x$, then $\frac{d}{dx}(f(\ln x)) =$ chain rule

$\boxed{\text{(A)}}$ $\frac{2\ln x + 2}{x}$ (B) $2x\ln x + 2x$ (C) $2\ln x + 2$ (D) $2\ln x + \frac{2}{x}$ (E) $\frac{2x+2}{x}$

$\frac{d}{dx}(f(\ln x)) = \frac{1}{x} \cdot f'(\ln x)$

$= \frac{1}{x} \cdot (2\ln x + 2)$

$= \frac{2\ln x + 2}{x}$

$f(x) = x^2 + 2x$

$f'(x) = 2x + 2$

$f'(\ln x) = 2\ln x + 2$

12. If $f(x) = e^{(2/x)}$, then $f'(x) =$

(A) $2e^{(2/x)}\ln x$ (B) $e^{(2/x)}$ (C) $e^{\left(-2/x^2\right)}$ $\boxed{\text{(D)}}$ $-\frac{2}{x^2}e^{(2/x)}$ (E) $-2x^2e^{(2/x)}$

$f(x) = e^{2x^{-1}}$

$f'(x) = -2x^{-2}e^{2x^{-1}}$

$= -\frac{2}{x^2}e^{2/x}$

Products, Quotients, and Composite Functions

a) For each of the functions $f(x)$ given below, decide whether the function is a product, a quotient, or a composition of simpler functions, and then complete the second and third columns of the table.

Example: $f(x) = x^3 \sec x$ Check one: ☑ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☐ Composition: $u(v(x))$	$u(x) = x^3$	$v(x) = \sec x$
1. $f(x) = \cos(\ln x)$ outside inside Check one: ☐ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☒ Composition: $u(v(x))$	$u(x) = \cos x$	$v(x) = \ln x$
2. $f(x) = \tan^{-1}(\sqrt{x})$ outside inside Check one: ☐ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☒ Composition: $u(v(x))$	$u(x) = \tan^{-1} x$	$v(x) = \sqrt{x}$
3. $f(x) = \frac{\sin x}{1 + \sin x}$ hi lo Check one: ☐ Product: $u(x) \cdot v(x)$ ☒ Quotient: $\frac{u(x)}{v(x)}$ ☐ Composition: $u(v(x))$	$u(x) = \sin x$	$v(x) = 1 + \sin x$
4. $f(x) = \underbrace{e^x}_{f} \underbrace{\csc x}_{g}$ Check one: ☒ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☐ Composition: $u(v(x))$	$u(x) = e^x$	$v(x) = \csc x$

5. $f(x) = \underbrace{e}_{\text{outside}}^{\overbrace{\sin^{-1}x}^{\text{inside}}}$ Check one: ☐ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☒ Composition: $u(v(x))$	$u(x) = e^x$	$v(x) = \sin^{-1}x$
6. $f(x) = \frac{3^x}{3^x + x}$ hi / lo Check one: ☐ Product: $u(x) \cdot v(x)$ ☒ Quotient: $\frac{u(x)}{v(x)}$ ☐ Composition: $u(v(x))$	$u(x) = 3^x$	$v(x) = 3^x + x$
7. $f(x) = \underbrace{\sqrt[3]{x}}_{f}\underbrace{\ln x}_{g}$ Check one: ☒ Product: $u(x) \cdot v(x)$ ☐ Quotient: $\frac{u(x)}{v(x)}$ ☐ Composition: $u(v(x))$	$u(x) = \sqrt[3]{x}$	$v(x) = \ln x$

b) Find $f'(x)$ for problems #1, 2, 5, and 7.

① $f(x) = \cos(\ln x)$

$f'(x) = \frac{1}{x} \cdot -\sin(\ln x)$

$$\boxed{f'(x) = -\frac{\sin(\ln x)}{x}}$$

② $f(x) = \tan^{-1}(\sqrt{x})$

$= \tan^{-1}(x^{1/2})$

$f'(x) = \frac{1}{2}x^{-1/2} \cdot \frac{1}{1 + (x^{1/2})^2}$

$= \frac{1}{2\sqrt{x}} \cdot \frac{1}{1+x}$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}(1+x)}}$$

⑤ $f(x) = e^{\sin^{-1}x}$

$f'(x) = \frac{1}{\sqrt{1-x^2}} \cdot e^{\sin^{-1}x}$

$$\boxed{f'(x) = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}}$$

⑦ $f(x) = \sqrt[3]{x} \ln x$

$= x^{1/3} \ln x$

$f'(x) = \ln x \cdot \frac{1}{3}x^{-2/3} + x^{1/3} \cdot \frac{1}{x}$

$= \frac{1}{3}x^{-2/3}\ln x + x^{-2/3}$

$$\boxed{f'(x) = x^{-2/3}\left(\frac{1}{3}\ln x + 1\right)}$$

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$x \quad y$

$y - y_1 = m(x - x_1)$

$\boxed{y - 1 = \frac{1}{4}(x + 1)}$

$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{3(1)^2 - (-1)}$

$= \frac{1}{3+1}$

$= \frac{1}{4}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$\frac{dy}{dx}$ DNE

$\frac{dy}{dx} = \frac{y}{3y^2 - x}$

$\frac{dy}{dx}$ DNE when $3y^2 - x = 0$

$3y^2 = x$

$y^3 - xy = 2$

$y^3 - (3y^2)y = 2$

$y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$\sqrt[3]{y^3} = \sqrt[3]{-1}$

$y = -1$

$y^3 - xy = 2$

$(-1)^3 - x(-1) = 2$

$-1 + x = 2$

$x = 3$

$\boxed{\text{Tangent line to curve is vertical at pt. } (3, -1)}$

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Continue problem 6 on page 21.

(c) Evaluate $\underbrace{\frac{d^2y}{dx^2}}_{\text{2nd derivative}}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

quotient rule... and implicit

$$\frac{d^2y}{dx^2} = \frac{(3y^2-x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}$$

from part a, $\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{4}$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,1)} = \frac{(3(1)^2-(-1))\cdot\frac{1}{4} - 1\left(6\cdot1\cdot\frac{1}{4}-1\right)}{(3(1)^2-(-1))^2}$$

$$= \frac{(3+1)\frac{1}{4} - 1\left(\frac{3}{2}-1\right)}{(3+1)^2}$$

$$= \frac{4\cdot\frac{1}{4} - 1\left(\frac{1}{2}\right)}{4^2}$$

$$= \frac{1-\frac{1}{2}}{16}$$

$$= \frac{\frac{1}{2}}{16}$$

$$= \frac{1}{2}\cdot\frac{1}{16}$$

$$\boxed{\left.\frac{d^2y}{dx^2}\right|_{(-1,1)} = \frac{1}{32}}$$

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GO ON TO THE NEXT PAGE.

AP Multiple-Choice Inverse Functions & Inverse Trigonometric Functions

$\nearrow y'(\frac{1}{4})$ chain

26. What is the slope of the line tangent to the curve $y = \arctan(4x)$ at the point at which $x = \frac{1}{4}$?

$\boxed{\text{(A)}}$ 2 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$ (E) -2

$$y' = 4 \cdot \frac{1}{1+(4x)^2}$$

$$y'(\tfrac{1}{4}) = 4 \cdot \frac{1}{1+(4\cdot\frac{1}{4})^2} = 4 \cdot \frac{1}{1+1^2} = 4 \cdot \frac{1}{1+1} = 4 \cdot \frac{1}{2} = 2$$

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inverses

27. Let f be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

(A) $\frac{1}{13}$ $\boxed{\text{(B)}}$ $\frac{1}{4}$ (C) $\frac{7}{4}$ (D) 4 (E) 13

$f'(x) = 3x^2 + 1$

$f'(1) = 3(1)^2 + 1 = 4$

switch x + y

$f(1) = 2$ $g(2) = 1$

same same

$f'(1) = 4$ $g'(2) = \frac{1}{4}$

reciprocal

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inverses

20. Let $f(x) = (2x+1)^3$ and let g be the inverse function of f. Given that $f(0) = 1$, what is the value of $g'(1)$?

(A) $-\frac{2}{27}$ (B) $\frac{1}{54}$ (C) $\frac{1}{27}$ $\boxed{\text{(D)}}$ $\frac{1}{6}$ (E) 6

$f(x) = (2x+1)^3$

$f'(x) = 2 \cdot 3(2x+1)^2$

$= 6(2x+1)^2$

$f'(0) = 6(2\cdot 0+1)^2 = 6(1)^2 = 6$

switch x + y

$f(0) = 1$ $g(1) = 0$

same same

$f'(0) = 6$ $g'(1) = \frac{1}{6}$

reciprocal

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28. Let f be a differentiable function such that $f(3) = 15$, $\boxed{f(6) = 3}$, $f'(3) = -8$, and $\boxed{f'(6) = -2}$. The function g is differentiable and $\underbrace{g(x) = f^{-1}(x)}_{\text{inverses}}$ for all x. What is the value of $g'(3)$?

$\boxed{\text{(A)}}$ $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

switch x + y

$f(6) = 3$ $g(3) = 6$

same same

$f'(6) = -2$ $g'(3) = -\frac{1}{2}$

reciprocal